FRÉCHET DIFFERENTIABILITY OF UNSTEADY INCOMPRESSIBLE NAVIER-STOKES FLOW WITH RESPECT TO DOMAIN VARIATIONS OF LOW REGULARITY BY USING A GENERAL ANALYTICAL FRAMEWORK *

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Abstract. We consider shape optimization problems governed by the unsteady Navier-Stokes equations by applying the method of mappings, where the problem is transformed to a reference domain Ω_{ref} and the physical domain is given by $\Omega = \tau(\Omega_{\text{ref}})$ with a domain transformation $\tau \in W^{1,\infty}(\Omega_{\text{ref}})$. We show the Fréchet-differentiability of $\tau \mapsto (v,p)(\tau)$ in a neighborhood of $\tau = \text{id}$ under as low regularity requirements on Ω_{ref} and τ as possible. We propose a general analytical framework beyond the implicit function theorem to show the Fréchet-differentiability of the transformation-to-state mapping conveniently. It can be applied to other shape optimization or optimal control problems and takes care of the usual norm discrepancy needed for nonlinear problems to show differentiability of the state equation and invertibility of the linearized operator. By applying the framework to the unsteady Navier-Stokes equations, we show that for Lipschitz domains Ω_{ref} and arbitrary $r > 1$, $s > 0$ the mapping $\tau \in (\mathbf{W}^{1,\infty} \cap \mathbf{W}^{1+s,r})(\Omega_{\text{ref}}) \mapsto (v,p)(\tau) \in (W(0,T;V) + W(0,T;\mathbf{H}_0^1)) \times (L^2(0,T;L_0^2) + W^{1,1}(0,T;\text{cl}_{(H^1)^*}(L_0^2))^*)$ is Fréchet-differentiable at $\tau = \text{id}$ and the mapping $\tau \in (\mathbf{W}^{1,\infty} \cap \mathbf{W}^{1+s,r})(\Omega_{\text{ref}}) \mapsto (v,p)(\tau) \in (L^2(0,T;\mathbf{H}_0^1) \cap C([0,T];\mathbf{L}^2) \times (L^2(0,T;L_0^2) + W^{1,1}(0,T;\text{cl}_{(H^1)^*}(L_0^2))^*)$ is Fréchet-differentiable on a neighborhood of id, where $V \subset \mathbf{H}_0^1(\Omega_{\text{ref}})$ is the subspace of solenoidal functions and $W(0,T;V)$ is the usual space of weak solutions. A crucial role in the analysis plays the handling of the incompressibility condition and the low time regularity of the pressure for weak solutions.

Key words. shape optimization, method of mappings, unsteady Navier-Stokes equations, Fréchet differentiability, PDE-constrained optimization

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1. Introduction.

In this paper, we analyse the Fréchet differentiability properties of the velocity field and the corresponding pressure component of viscous incompressible unsteady Navier-Stokes flows with respect to domain variations. Our aim is to keep the required regularity of the reference domain and the domain variations as low as possible. The setting includes, e.g., optimum shape design of a body exposed to Navier-Stokes flow. The results provide a rigorous analytical framework for the application of derivative- and adjoint-based optimization methods.

We use the approach of transformation to a reference domain, called method of mappings, to formulate the optimization problem on a fixed domain. The method of mappings goes back to Murat and Simon [32, 34] and is well suited for deriving rigorous Fréchet differentiability results with respect to domain variations. It also forms the basis of the proofs of many concepts in shape differential calculus. Moreover, it provides a suitable foundation for numerical implementations, cf. [7, 8]. The special case where the shape of the design boundary is described by the graph of a function was investigated in, e.g., [2, 27, 28]. The current paper investigates Fréchet

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differentiability of the state w.r.t. domain variations. This can be used, via the chain rule, for proving Fréchet differentiability of an objective function w.r.t. domain variations, but it is also of value independently of an objective function, e.g. for sensitivity analyses of the state, or in the context of all-at-once approaches for shape optimization, or when shape derivatives of multiple state-dependent functionals are required. The different, but related question of finding general conditions for proving the directional shape differentiability of a state-dependent objective function without using the chain rule is discussed in [24, 25] and in [29, 42], where in the latter the advantages of the distributed shape derivative are also highlighted and an averaged adjoint approach is proposed. These papers explore the fact that functionals with a special structure sometimes can be shape differentiable even if the state itself is not.

The incompressible Stokes and Navier-Stokes equations include the condition $\operatorname{div} \widetilde{v} = 0$ on Ω, where $\widetilde{v}$ denotes the velocity field. In the analysis of the Navier-Stokes equations it is usually built directly into the function space setting by choosing solenoidal (i.e., divergence-free) spaces. We use the tilde "$\widetilde{\ }$" here to indicate that $\widetilde{v}$ lives on Ω to distinguish it from $v(x) = \widetilde{v}(\tau(x))$, the pullback of $\widetilde{v}$ onto Ω_{ref} via the transformation $\Omega_{\text{ref}} \ni x \mapsto \widetilde{x} = \tau(x) \in \Omega$. If $\operatorname{div} \widetilde{v} = 0$ a.e. on Ω, then in general $\operatorname{div} v$ is *not* zero a.e. on Ω. Thus, divergence-free spaces are not mapped to divergence-free spaces. As we will see, this causes difficulties, especially in the unsteady case due to the low time regularity of the pressure. An alternative would be to use the Piola transform [10, sec. 1.7], which preserves divergence-freeness, but introduces additional nonlinearity and would require higher regularity requirements on τ, since it involves first derivatives of τ. In the case of the stationary Navier-Stokes equations there are further possibilities to surmount this difficulty. In [38], Simon uses a variant of the implicit function theorem to show Fréchet differentiability w.r.t. transformations for stationary Stokes flow when Ω is a $W^{2,\infty}$-domain. Bello, Fernández-Cara, and Simon [3, 4] showed Fréchet differentiability of the drag in the case of stationary Navier-Stokes equations when Ω is a $W^{2,\infty}$-domain; Bello, Fernández-Cara, Lemoine, and Simon [5] extended this result to Lipschitz domains under $W^{1,\infty}$-transformations. To treat the incompressibility condition, Bello et al. introduced in [3, 4] a family of isomorphisms to rewrite the equation $\operatorname{div}(v \circ \tau) = 0$ appropriately. In [5], Bello et al. state the incompressibility condition explicitly as we will do in this paper. For $m \geq 2$ the directional differentiability of the velocity and pressure field in $H^m \times H^{m-1}$ for C^{m+2}-variations of a C^{m+1}-domain was shown in [6].

In the context of the unsteady Navier-Stokes equations additional complications are caused by the fact that under standard regularity assumptions on the data the time regularity of the pressure and of the time derivative of the velocity is very low. Therefore we have to impose more regularity on the data to get stronger state spaces. As a consequence, for the linearized equation it is not clear how to choose the image space of the corresponding operator such that it defines an isomorphism as needed to apply the implicit function theorem as used in [5]. However, we show in this paper that using more regularity for the data and the slighty stronger space $\tau \in W^{1,\infty} \cap W^{1+s,r}$ for the transformations with arbitrary $r > 1$, $s > 0$, for Lipschitz domains Ω_{ref} the mapping $\tau \in (\mathbf{W}^{1,\infty} \cap \mathbf{W}^{1+s,r})(\Omega_{\text{ref}}) \mapsto (v,p)(\tau) \in (W(0,T;V) + W(0,T;\mathbf{H}_0^1)) \times (L^2(0,T;L_0^2) + W^{1,1}(0,T;\text{cl}_{(H^1)^*}(L_0^2))^*)$ is Fréchet-differentiable at $\tau = \text{id}$ and the mapping $\tau \in (\mathbf{W}^{1,\infty} \cap \mathbf{W}^{1+s,r})(\Omega_{\text{ref}}) \mapsto (v,p)(\tau) \in (L^2(0,T;\mathbf{H}_0^1) \cap C([0,T];\mathbf{L}^2) \times (L^2(0,T;L_0^2) + W^{1,1}(0,T;\text{cl}_{(H^1)^*}(L_0^2))^*)$ is Fréchet-differentiable on a neighborhood of id. Since the implicit function theorem is not applicable, we propose a general analytical framework beyond the implicit function theorem to show the Fréchet-differentiability of the transformation-to-state mapping in a systematic and convenient way. It can be applied to other choices of spaces as well as other shape optimization or optimal control problems and takes care of the usual norm discrepancy needed for nonlinear problems to show differentiability of the state equation and invertibility of the linearized operator. When applying this framework to shape optimization of the unsteady Navier-Stokes equations, we will handle the nonhomogeneous divergence condition in the linearized Navier-Stokes equations on the reference domain by using Bogovskiĭ's operator. This allows us to treat inhomogeneities of the divergence condition in $L^2(I; L^2_0(\Omega)) \cap H^1(I; (H^{3/2-\varepsilon})^*)$ on Lipschitz domains. For C^2-domains the Navier-Stokes equations with nonhomogeneous divergence condition in $L^2(I; H^{s-1/2}(\Omega)) \cap H^s(I; H^{-1/2}(\Omega))$ for $s > 1/2$ have been considered recently by Raymond [37].

Shape derivatives for the unsteady Navier-Stokes equations have been calculated formally e.g. in Pironneau [35, 36], and have been used in many applications, e.g. [31]. The existence of the weak Piola material and the shape derivative, i.e. the weak directional differentiability of the pull back of the velocity field and the velocity field, respectively, with respect to C^2-perturbations of C^2-domains has been shown in Gao et al. [15] for time independent domains and in [12], see also [13], by using the Piola transform and the speed method. Moreover, the boundary representation of the shape gradient according to the Hadamard-Zolésio structure theorem [11, 40] has been derived for several objective functions. In [39] Sokolowski and Stebel have proven the existence of the material derivative and thus the directional differentiability of the pull back of the velocity field with respect to C^2-perturbations of $C^{2,1}$-domains for unsteady incompressible non-Newtonian flows. Moreover, they derive the boundary representation of the shape gradient of an averaged drag functional.

The present paper focuses on the Fréchet differentiability of the pull back of velocity and pressure with respect to $W^{1,\infty} \cap W^{1+s,r}$-perturbations of $C^{0,1}$-domains and leads to Fréchet differentiability results for objective functionals, in particular averaged drag functionals. This yields a rigorous foundation for derivative-based optimization methods. The differentiability results that we develop in this paper provide a rigorous basis for computing shape derivatives of an objective function using, e.g., the adjoint method for which, in a notation similar to the one used in this paper, details can be found in [7, sec. 3.4 and 3.5] and [30, sec. 3.3].

The presented techniques and especially Section 3 can also be applied to other classes of nonlinear PDE systems. In fact, very recently, the techniques developed in the current paper have also successfully been applied to the instationary Boussinesq equations with Dirichlet conditions for the velocity and inhomogeneous Robin boundary conditions for the temperature, cf. the first author's submitted PhD thesis [14].

The paper is organized as follows. In Section 2 we recall the method of mappings by Murat and Simon in the abstract setting and apply it to the unsteady Navier-Stokes equations. Furthermore we state some regularity results for the solutions of the Navier-Stokes equations. In Section 3 we introduce an abstract differentiability theorem, which provides a convenient framework to prove the Fréchet differentiability of the solution operator of nonlinear optimal control problems. In Section 4 we apply this framework to shape optimization of the Navier-Stokes equations and show Fréchet differentiability of the solution with respect to domain transformations in a suitable Banach space framework. In Section 5 the results from Section 4 are used to prove the Fréchet differentiability of a wide class of reduced objective functions in a neighbourhood of the identity. Furthermore, the results are applied to practical objective functions for the minimization of the mean drag of a body and for vortex reduction in viscous flows.

2. Shape optimization for the unsteady Navier-Stokes equations.

2.1. Shape optimization via the mapping method. Consider a shape optimization problem of the general form

$$\min \quad \widetilde{J}(\widetilde{y}, \Omega) \quad \text{s.t.} \quad \widetilde{E}(\widetilde{y}, \Omega) = 0, \quad \Omega \in \mathcal{O}_{\text{ad}}. \tag{2.1}$$

Here, $\Omega \subset \mathbb{R}^d$ is the domain to be optimized and $\mathcal{O}_{\text{ad}}$ is the set of admissible domains. The state $\widetilde{y}$ lives on $\Omega \times I$, $I = (0, T)$, and satisfies the state equation $\widetilde{E}(\widetilde{y}, \Omega) = 0$, in our case the unsteady Navier-Stokes equations on Ω with initial and boundary conditions. We assume that the admissible shapes can be described by

$$\mathcal{O}_{\text{ad}} = \{\tau(\Omega_{\text{ref}}) \,:\, \tau \in T_{\text{ad}}\},$$

where T_{ad} is a set of admissible transformations and $\Omega_{\text{ref}} \subset \mathbb{R}^d$ is a bounded reference domain with Lipschitz boundary. Hence, admissible domains $\Omega \in \mathcal{O}_{\text{ad}}$ are interpreted as images of Ω_{ref} under suitable transformations $\tau : \mathbb{R}^d \to \mathbb{R}^d$, $d = 2$ or 3. Our minimum requirements that will be slightly strengthened later are that $\tau : \Omega_{\text{ref}} \to \tau(\Omega_{\text{ref}})$ is invertible, $\tau(\Omega_{\text{ref}})$ is a bounded Lipschitz domain and $\tau \in W^{1,\infty}(\Omega_{\text{ref}})^d$, $\tau^{-1} \in W^{1,\infty}(\tau(\Omega_{\text{ref}}))^d$.

For functions $\widetilde{w}$ defined on $\Omega \times I$, we introduce the pullback to Ω_{ref} by

$$w := \widetilde{w} \circ_{\widetilde{x}} \tau, \quad w(x,t) = \widetilde{w}(\tau(x), t).$$

This is well defined if $\widetilde{w}$ is L^1 in space. We also write $w(t) = w(\cdot, t) = \widetilde{w}(t) \circ \tau$. Similarly, if $\widetilde{w}$ is defined on Ω, we set $w := \widetilde{w} \circ \tau$, $w(x) = \widetilde{w}(\tau(x))$.

Let now $y = \widetilde{y} \circ_{\widetilde{x}} \tau$ be the pullback of $\widetilde{y}$ to Ω_{ref}. We abbreviate the variational form of the state equation by

$$\langle \widetilde{E}(\widetilde{y}, \Omega), \widetilde{\varphi} \rangle_\Omega = 0 \quad \forall\, \widetilde{\varphi} \in \Phi(\Omega), \tag{2.2}$$

where $\Phi(\Omega)$ is a suitable space of test functions. Since τ^{-1} exists, we can write $y = \widetilde{y} \circ_{\widetilde{x}} \tau$ and $\varphi = \widetilde{\varphi} \circ_{\widetilde{x}} \tau$ also in the equivalent form $\widetilde{y} = y \circ_x \tau^{-1}$ and $\widetilde{\varphi} = \varphi \circ_x \tau^{-1}$.

Next, we define $E(y, \tau)$ via

$$\langle E(\widetilde{y} \circ_x \tau, \tau), \widetilde{\varphi} \circ_{\widetilde{x}} \tau \rangle_{\Omega_{\text{ref}}} := \langle \widetilde{E}(\widetilde{y}, \Omega), \widetilde{\varphi} \rangle_\Omega \quad \forall\, \widetilde{y} \in Y(\Omega), \;\; \forall\, \widetilde{\varphi} \in \Phi(\Omega), \;\; \forall\, \tau \in T_{\text{ad}}.$$

Assumption 2.1. *Assume that we can choose Y_{ref} and Φ_{ref} such that for all $\tau \in T_{\text{ad}}$ and $\Omega = \tau(\Omega_{\text{ref}})$, there holds:*

1. $\Phi_{\text{ref}} = \{\widetilde{\varphi} \circ_{\widetilde{x}} \tau \,:\, \widetilde{\varphi} \in \Phi(\Omega)\}$
2. $Y_{\text{ref}} = \{\widetilde{y} \circ_{\widetilde{x}} \tau \,:\, \widetilde{y} \in Y(\Omega)\}$
3. *the solution $\widetilde{y}(\Omega)$ of* (2.2) *satisifies $\widetilde{y}(\Omega) \circ_{\widetilde{x}} \tau \in Y_{\text{ref}}$.*

Under Assumption 2.1 the transformed state equation in variational form reads

$$\langle E(y, \tau), \varphi \rangle_{\Omega_{\text{ref}}} = 0 \quad \forall\, \varphi \in \Phi_{\text{ref}}. \tag{2.3}$$

It is equivalent to (2.2) in the sense that $\widetilde{y} = \widetilde{y}(\Omega)$, with $\Omega = \tau(\Omega_{\text{ref}})$, solves (2.2) if and only if $y(\tau) := \widetilde{y}(\Omega) \circ_{\widetilde{x}} \tau \in Y_{\text{ref}}$ solves (2.3). Conversely, $y(\tau) \in Y_{\text{ref}}$ solves (2.3) if and only if $\widetilde{y} = \widetilde{y}(\Omega) := y(\tau) \circ_x \tau^{-1}$ solves (2.2) with $\Omega = \tau(\Omega_{\text{ref}})$.

Thus, with $J(y,\tau) := \widetilde{J}(y \circ_x \tau^{-1}, \tau(\Omega_{\text{ref}}))$ for all $\tau \in T_{\text{ad}}$ and $y \in Y_{\text{ref}}$, we can rewrite (2.1) on the fixed domain Ω_{ref} as follows:

$$\min \quad J(y,\tau) \quad \text{s.t.} \quad E(y,\tau) = 0, \quad \tau \in T_{\text{ad}}. \tag{2.4}$$

Usually, the concrete variational form of the state equation on the reference domain is obtained by using the transformation rule for integrals and for derivatives. This will be carried out for the unsteady Navier-Stokes equations in the following. By convention, we denote all quantities on the physical domains $\tau(\Omega_{\text{ref}})$ by $\widetilde{\ }$.

2.2. The unsteady Navier-Stokes equations. Now we apply the presented method of mappings to shape optimization problems governed by the unsteady Navier-Stokes equations for a viscous, incompressible fluid on a bounded domain $\Omega = \tau(\Omega_{\text{ref}}) \subset \mathbb{R}^d$ with Lipschitz boundary. To avoid technicalities in the formulation of the equations, we consider homogeneous Dirichlet boundary conditions. The analysis can, however, be extended also to the inhomogeneous case and to other types of boundary conditions. We then would have to require that a neighborhood of that part of the boundary where inhomogeneous Dirichlet conditions are posed remains pointwise fixed for all admissibe transformations. We thus consider the problem

$$\begin{aligned} \widetilde{v}_t - \nu\Delta\widetilde{v} + (\widetilde{v}\cdot\nabla)\widetilde{v} + \nabla\widetilde{p} &= \widetilde{f} && \text{on } \Omega \times I, \\ \operatorname{div}\widetilde{v} &= 0 && \text{on } \Omega \times I \\ \widetilde{v} &= 0 && \text{on } \partial\Omega \times I, \\ \widetilde{v}(\cdot,0) &= \widetilde{v}_0 && \text{on } \Omega \end{aligned} \tag{2.5}$$

where $\widetilde{v} : \Omega \times I \to \mathbb{R}^d$ denotes the velocity and $\widetilde{p} : \Omega \times I \to \mathbb{R}$ is the pressure of the fluid. Here $I = (0,T)$, $T > 0$, is the time interval, $\nu > 0$ is the kinematic viscosity and $\nabla\widetilde{v}(\widetilde{x},t) = \widetilde{v}_{\widetilde{x}}(\widetilde{x},t)^T$ denotes the transpose of the spatial Jacobian matrix of $\widetilde{v}$.

We use bold face notation for $\mathbb{R}^d$-valued standard function spaces, e.g., $\mathbf{L}^q(\Omega) := L^q(\Omega)^d$, $\mathbf{H}_0^1(\Omega) := H_0^1(\Omega)^d$, $\mathbf{H}^{-1}(\Omega) := H^{-1}(\Omega)^d$ and $\mathbf{H}_0^{-1}(\Omega) := (H^1(\Omega)^*)^d$ etc., and define the spaces

$$\mathcal{V}(\Omega) := \{\widetilde{v} \in \mathbf{C}_0^\infty(\Omega) : \operatorname{div}\widetilde{v} = 0\}, \quad V(\Omega) := \operatorname{cl}_{\mathbf{H}_0^1(\Omega)}(\mathcal{V}(\Omega)),$$
$$H(\Omega) := \operatorname{cl}_{\mathbf{L}^2(\Omega)}(\mathcal{V}(\Omega)), \quad L_0^2(\Omega) := \{\widetilde{p} \in L^2(\Omega) : \int_\Omega \widetilde{p} = 0\}.$$

We note that $L_0^2(\Omega)$ can be identified with $L^2(\Omega)/\mathbb{R}$, where

$$\|\widetilde{p}\|_{L^2(\Omega)/\mathbb{R}} := \inf_{c\in\mathbb{R}} \|\widetilde{p} + c\|_{L^2(\Omega)}.$$

In fact, we have $\|\widetilde{p}\|_{L_0^2(\Omega)} = \|\widetilde{p}\|_{L^2(\Omega)/\mathbb{R}}$ for all $\widetilde{p} \in L_0^2(\Omega)$.

It will be convenient to introduce the projection

$$P_0(\Omega) \in L(L^2(\Omega), L_0^2(\Omega)), \quad P_0(\Omega)\tilde{p} = \tilde{p} - \frac{\int_\Omega \tilde{p}(x)\,dx}{\int_\Omega 1\,dx}. \tag{2.6}$$

The Bochner spaces of r-integrable X-valued functions are denoted by $L^r(I;X)$. We use the usual norms on the spaces $L^r(\Omega)$ and $L^r(I;X)$ and the inner products

$$(\widetilde{v},\widetilde{w})_{H(\Omega)} := (\widetilde{v},\widetilde{w})_{\mathbf{L}^2(\Omega)}, \quad (\widetilde{v},\widetilde{w})_{V(\Omega)} := (\widetilde{v},\widetilde{w})_{\mathbf{H}_0^1(\Omega)} = \sum_{i=1}^d (\widetilde{v}_{x_i},\widetilde{w}_{x_i})_{\mathbf{L}^2(\Omega)}.$$

Moreover, let $V(\Omega) \hookrightarrow H(\Omega) = H(\Omega)^* \hookrightarrow V(\Omega)^*$ denote the corresponding Gelfand triple and define

$$W(I;V(\Omega)) := \{\widetilde{v} \in L^2(I;V(\Omega)) : \ \widetilde{v}_t \in L^2(I;V(\Omega)^*)\}.$$

Here, $\widetilde{v}_t$ denotes the distributional derivative. Note that $\widetilde{v} \in L^2(I;V(\Omega))$ implies $\widetilde{v} \in L^2(I;L^2(\Omega))$ and thus one has also $\widetilde{v}_t \in H^{-1}(I;L^2(\Omega))$. In the same way, the space $W(I;\mathbf{H}_0^1(\Omega))$ used later is defined based on the Gelfand triple $H_0^1(\Omega) \hookrightarrow L^2(\Omega) = L^2(\Omega)^* \hookrightarrow H^{-1}(\Omega)$.

Let $\widetilde{f} \in L^2(I;H^{-1}(\Omega))$. The weak formulation of (2.5) for the velocity field is: Find $\widetilde{v} \in W(I;V(\Omega))$ such that

$$\begin{aligned}(2.7)\qquad &\langle \widetilde{v}_t(t), \widetilde{w}\rangle_{H^{-1}(\Omega),H_0^1(\Omega)} + \nu(\nabla\widetilde{v}(t), \nabla\widetilde{w})_{\mathbf{L}^2(\Omega)} + \int_\Omega [(\widetilde{v}(t)\cdot\nabla)\widetilde{v}(t)]\cdot\widetilde{w}\,d\widetilde{x} \\ &\qquad = \langle \widetilde{f}(t), \widetilde{w}\rangle_{H^{-1}(\Omega),H_0^1(\Omega)} \quad \forall\,\widetilde{w} \in V(\Omega) \text{ and a.a. } t \in I, \\ &\widetilde{v}(\cdot,0) = \widetilde{v}_0 \text{ in } H(\Omega).\end{aligned}$$

It is well known that so far the question of existence and uniqueness is answered satisfactorily only in the case $d \le 2$. In fact for $d = 2$, there holds

PROPOSITION 2.1. *Let $d = 2$ and assume*

$$(2.8)\qquad \widetilde{f} \in L^2(I;H^{-1}(\Omega)), \quad \widetilde{v}_0 \in H(\Omega).$$

Then there exists a unique weak solution $\widetilde{v}$ of (2.5) according to (2.7). It satisfies

$$(2.9)\qquad \widetilde{v} \in W(I;V(\Omega)) \hookrightarrow C(\bar{I};H(\Omega)).$$

The pressure $\widetilde{p}$ can be introduced as a distribution on $\Omega \times (0,T)$.

For a proof see e.g. [43, Ch. III].

A weak solution $\widetilde{v} \in W(I;V(\Omega))$ can equivalently be characterized by

(2.10)

$$\begin{aligned}&-(\widetilde{v}, \widetilde{w}_t)_{\mathbf{L}^2(\Omega\times I)} - (\widetilde{v}_0, \widetilde{w}(0))_{\mathbf{L}^2(\Omega)} + \nu(\nabla\widetilde{v}, \nabla\widetilde{w})_{\mathbf{L}^2(\Omega\times I)} + \int_0^T\int_\Omega [(\widetilde{v}\cdot\nabla)\widetilde{v}]\cdot\widetilde{w}\,d\widetilde{x}\,dt \\ &= \int_0^T \langle \widetilde{f}(t), \widetilde{w}(t)\rangle_{V(\Omega)^*,V(\Omega)}\,dt \quad \forall\,\widetilde{w} \in C_0^\infty(\Omega\times[0,T)),\ \operatorname{div}(\widetilde{w}(t)) = 0\ \forall\,t \in [0,T).\end{aligned}$$

If the data $\widetilde{f}$ and $\widetilde{v}_0$ are more regular, then the solution has further regularity.

PROPOSITION 2.2. *Let $d = 2$ and assume*

$$(2.11)\qquad \widetilde{f} \in H^1(I;\mathbf{H}^{-1}(\Omega)), \quad \widetilde{f}(0,\cdot) \in \mathbf{L}^2(\Omega), \quad \widetilde{v}_0 \in V(\Omega)\cap\mathbf{H}^2(\Omega).$$

Then the weak solution $(\widetilde{v},\widetilde{p})$ of the Navier-Stokes equations satisfies

$$(2.12)\qquad \widetilde{v} \in C(\bar{I};V(\Omega)), \quad \widetilde{v}_t \in L^2(I,V(\Omega))\cap L^\infty(I;H(\Omega)), \quad \widetilde{p} \in L^\infty(I;L_0^2(\Omega))$$

Furthermore there exists a constant $c(\Omega) > 0$ with

$$\begin{aligned}(2.13)\qquad \|\widetilde{p}\|_{L^\infty(I;L_0^2(\Omega))} \le c(\Omega)(&\|\widetilde{v}_t\|_{L^\infty(I;H(\Omega))} + \nu\|\widetilde{v}\|_{L^\infty(I;V(\Omega))} + \\ &\|\widetilde{v}\|^2_{L^\infty(I;V(\Omega))} + \|\widetilde{f}\|_{L^\infty(I;\mathbf{H}^{-1}(\Omega))})\end{aligned}$$

Proof. The regularity results for the velocity can be found in [43, Thm. III.3.5]. To study the pressure we use $H_0^1(\Omega) \hookrightarrow L^4(\Omega)$, $L^{4/3}(\Omega) \hookrightarrow H^{-1}(\Omega)$ and obtain

$$\|(\widetilde{v}\cdot\nabla)\widetilde{v}\|_{\mathbf{H}^{-1}(\Omega)} \le c_1\|(\widetilde{v}\cdot\nabla)\widetilde{v}\|_{\mathbf{L}^{4/3}(\Omega)} \le c_2\|\widetilde{v}\|_{\mathbf{L}^4(\Omega)}\|\nabla\widetilde{v}\|_{\mathbf{L}^2(\Omega)} \le c_3\|\widetilde{v}\|^2_{V(\Omega)}$$

with constants c_1, c_2, c_3. Hence, introducing $A(\widetilde{v})(t) := \nu(\nabla\widetilde{v}(t), \nabla\cdot)_{\mathbf{L}^2(\Omega)} \in \mathbf{H}^{-1}(\Omega)$ for a.a. $t \in I$ we have

$$\|(\widetilde{v}\cdot\nabla)\widetilde{v}\|_{L^\infty(I;\mathbf{H}^{-1}(\Omega))} \le c_3\|\widetilde{v}\|^2_{L^\infty(I,V(\Omega))}, \quad \|A(\widetilde{v})\|_{L^\infty(I;\mathbf{H}^{-1}(\Omega))} \le \nu\|\widetilde{v}\|_{L^\infty(I,V(\Omega))}.$$

Now $\widetilde{v}_t \in L^\infty(I;H(\Omega)) \hookrightarrow L^\infty(I;\mathbf{H}^{-1}(\Omega))$, since $H(\Omega) \hookrightarrow \mathbf{L}^2(\Omega) \hookrightarrow \mathbf{H}^{-1}(\Omega)$. Furthermore, $\widetilde{f} \in L^\infty(I;\mathbf{H}^{-1}(\Omega))$, since $\widetilde{f} \in H^1(I;\mathbf{H}^{-1}(\Omega))$ (see e.g. [43, Lem. III.1.1]). Hence,

$$g := -\widetilde{v}_t - (\widetilde{v}\cdot\nabla)\widetilde{v} - A(\widetilde{v}) + \widetilde{f} \in L^\infty(I;\mathbf{H}^{-1}(\Omega))$$

Using $\langle g, \widetilde{w}\rangle_{\mathbf{H}^{-1}(\Omega),\mathbf{H}_0^1(\Omega)} = 0$ for all $\widetilde{w} \in V(\Omega)$ and a.a. $t \in I$, there exists by [43, Rem. I.1.4] for a.a. t a unique $\widetilde{p}(t) \in L_0^2(\Omega)$ with $\nabla\widetilde{p}(t) = g(t)$, where $\|\widetilde{p}(t)\|_{L_0^2(\Omega)} \le c_4\|g(t)\|_{\mathbf{H}^{-1}(\Omega)}$ by [43, Prop. I.1.2] with a constant $c_4 > 0$. Hence there exists a unique pressure $\widetilde{p} \in L^\infty(I;L_0^2(\Omega))$ which satisfies (2.13) □

For the three-dimensional case, an existence and uniqueness result holds for example under the additional assumption

Assumption 2.2. *Let*

$$d_1(\Omega) := \|\widetilde{f}(0)\|_{\mathbf{L}^2(\Omega)} + \nu c_0\|\widetilde{v}_0\|_{\mathbf{H}^2(\Omega)} + c_1\|\widetilde{v}_0\|^2_{\mathbf{H}^2(\Omega)},$$

where c_0, c_1 are constants such that $\|\Delta v\|_{L^2(\Omega)} \le c_0\|v\|_{\mathbf{H}^2(\Omega)}$ for all $v \in \mathbf{H}^2(\Omega)$ and $|((u\cdot\nabla)u, v)_{\mathbf{L}^2(\Omega)}| \le c_1\|u\|_{\mathbf{H}_0^1(\Omega)}\|u\|_{\mathbf{H}^2(\Omega)}\|v\|_{\mathbf{L}^2(\Omega)}$ for all $u \in \mathbf{H}^2(\Omega), v \in \mathbf{L}^2(\Omega)$. Moreover, let $d_2(\Omega) := \|\widetilde{f}\|^2_{L^\infty(I;V(\Omega)^*)}$. We assume that there exists $c^*(\Omega)$ such that

$$\frac{d_2(\Omega)}{\nu} + (1 + d_1(\Omega)^2)\left(\|\widetilde{v}_0\|^2_{\mathbf{L}^2(\Omega)} + \frac{Td_2(\Omega)}{\nu}\right)^{\frac{1}{2}} e^{\|\widetilde{f}_t\|_{L^1(I;\mathbf{L}^2(\Omega))}} \le c^*(\Omega) < \frac{\nu^3}{c^2}.$$

Here c denotes a constant with $|((u\cdot\nabla)v, w)_{\mathbf{L}^2(\Omega)}| \le c\|u\|_{\mathbf{H}_0^1(\Omega)}\|v\|_{\mathbf{H}_0^1(\Omega)}\|w\|_{\mathbf{H}_0^1(\Omega)}$ for all $u, v, w \in \mathbf{H}_0^1(\Omega)$.

This assumption is satisfied if ν is large enough or if $\widetilde{f}$ and $\widetilde{v}_0$ are small enough.

Proposition 2.3. *Let $d = 3$ and assume*

$$\widetilde{f} \in W^{1,1}(I;\mathbf{L}^2(\Omega)),\ \widetilde{v}_0 \in V(\Omega)\cap\mathbf{H}^2(\Omega),\ \textit{Assumption 2.2.} \tag{2.14}$$

Then the solution $(\widetilde{v}, \widetilde{p})$ of the Navier-Stokes equations is unique and satisfies

$$\widetilde{v} \in C(\bar{I};V(\Omega)),\quad \widetilde{v}_t \in L^2(I,V(\Omega))\cap L^\infty(I;H(\Omega)),\quad \widetilde{p} \in L^\infty(I;L_0^2(\Omega)). \tag{2.15}$$

Proof. The part for the velocity can be found in [43, Thm. III.3.7]. The pressure regularity and uniqueness can be obtained in the very same way as in Proposition 2.2 because $\widetilde{f} \in W^{1,1}(I;\mathbf{L}^2(\Omega))$ implies $\widetilde{f} \in L^\infty(I;\mathbf{H}^{-1}(\Omega))$. □

2.3. Transformation to the reference domain.

2.3.1. Admissible transformations and notations. We will work under the following regularity assumptions on the domain transformations and the data.

ASSUMPTION 2.3. *$\Omega_{\text{ref}} \subset \mathbb{R}^d$ is a bounded Lipschitz domain. Moreover for arbitrary but fixed $1 < r < 2$ and $0 < s < 1/r$ let*

$$U = (\mathbf{W}^{1,\infty} \cap \mathbf{W}^{1+s,r})(\Omega_{\text{ref}})$$

and let $\mathcal{O}_{\text{ad}} = \{\tau(\Omega_{\text{ref}}) : \tau \in T_{\text{ad}}\}$, where with a constant $\delta > 0$

$$\begin{aligned} T_{\text{ad}} \subset \{\tau \in U :\ & \tau^{-1} \in (\mathbf{W}^{1,\infty} \cap \mathbf{W}^{1+s,r})(\tau(\Omega_{\text{ref}})), \\ & \tau(\Omega_{\text{ref}}) \text{ is a bounded Lipschitz-domain}, \ \operatorname{essinf}_{x\in\Omega_{\text{ref}}} \det(\tau'(x)) > \delta > 0\} \end{aligned}$$

is relatively open with respect to the topology of U (and thus in fact open in U by Lemma 4.1 below). Here, $\tau'(x) = \nabla\tau(x)^T$ denotes the Jacobian of τ.

Moreover, the data $\widetilde{v}_0, \widetilde{f}$ are given such that

$$\widetilde{f} \in L^\infty(I; \mathbf{C}^1(\overline{\Omega})),\ \widetilde{f}_t \in L^2(I; \mathbf{H}^{-1}(\Omega)),\ \widetilde{f}(0) \in H(\Omega),\ \widetilde{v}_0 \in V(\Omega) \cap \mathbf{H}^2(\Omega) \cap \mathbf{C}^1(\overline{\Omega})$$

for all $\Omega \in \mathcal{O}_{\text{ad}}$ and they are used on all $\Omega \in \mathcal{O}_{\text{ad}}$. For $d = 3$ we furthermore assume that $\widetilde{f} \in W^{1,1}(I; \mathbf{L}^2(\Omega))$ and that Assumption 2.2 holds true for all $\Omega \in \mathcal{O}_{\text{ad}}$.

REMARK 2.4. *Note that a bi-Lipschitzian image $\tau(\Omega_{\text{ref}})$ of a bounded Lipschitz domain Ω_{ref} is not always a bounded Lipschitz domain, see [19, 1.2], but for $\|\tau - \text{id}\|_{\mathbf{W}^{1,\infty}(\Omega_{\text{ref}})}$ small enough, this is ensured. The latter follows from [5, Lem. 3] and the fact that there exists a linear bounded extension operator $\mathbf{W}^{1,\infty}(\Omega_{\text{ref}}) \to \mathbf{W}^{1,\infty}(\mathbb{R}^d)$ for the bounded Lipschitz domain Ω_{ref}, see e.g. [41, Thm. 5, p. 181].*

In the following, we will use the following functions defined on T_{ad} to abbreviate the τ-dependent terms that result from the transformation to the reference domain.

$$g_1(\tau) := \det(\tau'),\ g_2(\tau) := \tau'^{-1}\tau'^{-T} g_1(\tau),\ g_3(\tau) := \tau'^{-T} g_1(\tau). \tag{2.16}$$

We note that these functions are Fréchet differentiable from $\mathbf{W}^{1,\infty}(\Omega_{\text{ref}})$ to $L^\infty(\Omega_{\text{ref}})$.

LEMMA 2.5. *Let Assumption 2.3 hold. Then the mappings*

$$g_1 : (T_{\text{ad}}, \|\cdot\|_{\mathbf{W}^{1,\infty}(\Omega_{\text{ref}})}) \to L^\infty(\Omega_{\text{ref}}),\ g_2, g_3 : (T_{\text{ad}}, \|\cdot\|_{\mathbf{W}^{1,\infty}(\Omega_{\text{ref}})}) \to L^\infty(\Omega_{\text{ref}})^{d\times d}$$

defined in (2.16) *are continuously Fréchet differentiable with derivatives*

$$\begin{aligned} g_1'(\tau)\psi &= \operatorname{tr}(\tau'^{-1}\psi') g_1(\tau), \\ g_2'(\tau)\psi &= -\tau'^{-1}(\psi'\tau'^{-1} + \tau'^{-T}\psi'^T - \operatorname{tr}(\tau'^{-1}\psi')I)\tau'^{-T} g_1(\tau), \\ g_3'(\tau)\psi &= -(\tau'^{-T}\psi'^T - \operatorname{tr}(\tau'^{-1}\psi')I)\tau'^{-T} g_1(\tau). \end{aligned}$$

For $\tau = \text{id}$ we have

$$g_1'(\text{id})\psi = \operatorname{tr}(\psi'), \quad g_2'(\text{id})\psi = -\psi' - \psi'^T + \operatorname{tr}(\psi')I, \quad g_3'(\text{id})\psi = -\psi'^T + \operatorname{tr}(\psi')I.$$

Proof. Since for all $\tau \in T_{\text{ad}}$ we have $g_1(\tau) \geq \delta > 0$ a.e., we know that τ' is invertible a.e. on Ω_{ref}. The differentiability and the formula for the derivative now follow from elementary pointwise arguments. ∎

Finally, we introduce the trilinear forms $b(u, v, w)$ and $\widehat{b}(u, v, w, M)$ to abbreviate the convection terms in the classical and later on in the transformed Navier-Stokes equations, respectively. To this end, we define for $u \in L^{r_u}(I; \mathbf{L}^{q_u}(\Omega_{\text{ref}}))$, $v \in$

$L^1(I;\mathbf{L}^1(\Omega_{\text{ref}}))$ with $\nabla v \in L^{r_v}(I;L^{q_v}(\Omega_{\text{ref}})^{d\times d})$, $w \in L^{r_w}(I;\mathbf{L}^{q_w}(\Omega_{\text{ref}}))$, $q_u, q_v, q_w \in [1,\infty]$, $\frac{1}{q_u}+\frac{1}{q_v}+\frac{1}{q_w}=1$, $r_u, r_v, r_w \in [1,\infty]$, $\frac{1}{r_u}+\frac{1}{r_v}+\frac{1}{r_w}$, and $M \in L^\infty(\Omega_{\text{ref}})^{d\times d}$:

$$
\begin{aligned}
b(u(t),v(t),w(t)) &:= \int_{\Omega_{\text{ref}}} [(u(t)\cdot\nabla)v(t)]\cdot w(t)\;dx,\\
b_I(u,v,w) &:= \int_0^T b(u(t),v(t),w(t))\,dt,
\end{aligned}
\tag{2.17}
$$

$$
\begin{aligned}
\widehat{b}(u(t),v(t),w(t),M) &:= \int_{\Omega_{\text{ref}}} [((M^T u(t))\cdot\nabla)v(t)]\cdot w(t)\;dx,\\
\widehat{b}_I(u,v,w,M) &:= \int_0^T \widehat{b}(u(t),v(t),w(t),M)\,dt.
\end{aligned}
\tag{2.18}
$$

LEMMA 2.6. *Let $u, v, w \in \mathbf{H}_0^1(\Omega_{\text{ref}})$ and $M \in L^\infty(\Omega_{\text{ref}})^{d\times d}$. Then with a constant $C = C(d)$ we have*

$$
\begin{aligned}
|b(u,v,w)| &\le C\|u\|_{\mathbf{L}^2}^{1-\frac{d}{4}}\|u\|_{\mathbf{H}_0^1}^{\frac{d}{4}}\|v\|_{\mathbf{H}_0^1}\|w\|_{\mathbf{L}^2}^{1-\frac{d}{4}}\|w\|_{\mathbf{H}_0^1}^{\frac{d}{4}},\\
|\widehat{b}(u,v,w,M)| &\le C\|u\|_{\mathbf{L}^2}^{1-\frac{d}{4}}\|u\|_{\mathbf{H}_0^1}^{\frac{d}{4}}\|v\|_{\mathbf{H}_0^1}\|w\|_{\mathbf{L}^2}^{1-\frac{d}{4}}\|w\|_{\mathbf{H}_0^1}^{\frac{d}{4}}\|M\|_{\mathbf{L}^\infty},
\end{aligned}
$$

for $d \in \{2,3\}$.

Proof. The estimate for b for $d = 2$ can be found in [43, Lem. III.3.4] and in slightly different form also for $d = 3$. Now the results for $\widehat{b}$ are easily obtained. □

2.3.2. Transformation of the Navier-Stokes equations to the reference domain. Since solenoidality is not preserved by the pullback operation, the spaces $\tau^* V(\Omega) := \{\widetilde{v}\circ\tau : \widetilde{v} \in V(\Omega)\}$ and $\tau^* H(\Omega)$, $\Omega = \tau(\Omega_{\text{ref}})$, would depend on τ. We avoid this by working with the spaces $\mathbf{H}_0^1(\Omega)$ and $\mathbf{L}^2(\Omega)$ instead of $V(\Omega)$ and $H(\Omega)$, reintroduce the pressure, and formulate the divergence-free condition explicitly. This results in the following weak velocity-pressure formulation of problem (2.7), where the divergence-freeness of the velocity is not included in the trial and test spaces. In view of a subsequent transformation to the reference domain, the time derivative of the velocity is required to have some additional regularity. To keep the required regularity for the transformations low, we assume that the weak solution satisfies

$$
\widetilde{v} \in L^2(I;V(\Omega)),\quad \widetilde{v}_t \in L^2(I;\mathbf{L}^2(\Omega)),\quad \widetilde{p} \in L^2(I;L_0^2(\Omega)).
\tag{2.19}
$$

Note that a better regularity than (2.19) is ensured if the data $\widetilde{v}_0$ and $\widetilde{f}$ satisfy the requirements of Proposition 2.2 and 2.3, respectively.

Under the above reguarity assumptions (2.19) together with $\widetilde{f} \in C(\bar{I};\mathbf{H}^{-1}(\Omega))$, the weak formulation (2.7) is equivalent to

$$
\begin{aligned}
\langle \widetilde{v}_t(t) + (\widetilde{v}(t)\cdot\nabla)\widetilde{v}(t), \widetilde{w}\rangle_{\mathbf{H}^{-1}(\Omega),\mathbf{H}_0^1(\Omega)} &+ \nu(\nabla\widetilde{v}(t),\nabla\widetilde{w})_{\mathbf{L}^2(\Omega)} - (\widetilde{p}(t), \operatorname{div}\widetilde{w})_{L_0^2(\Omega)}\\
&= \langle \widetilde{f}(t), \widetilde{w}\rangle_{\mathbf{H}^{-1}(\Omega),\mathbf{H}_0^1(\Omega)} \quad \forall\, \widetilde{w} \in \mathbf{H}_0^1(\Omega) \text{ for a.a. } t \in I,\\
(\widetilde{q}, \operatorname{div}\widetilde{v}(t))_{L_0^2(\Omega)} &= 0 \qquad\qquad\qquad\qquad\quad \forall\, \widetilde{q} \in L_0^2(\Omega) \text{ for a.a. } t \in I,\\
\widetilde{v}|_{t=0} &= \widetilde{v}_0.
\end{aligned}
\tag{2.20}
$$

REMARK 2.7. *The regularity requirement on $\widetilde{v}_t$ can be weakened if the time derivative is written in very weak form, i.e., with a time-dependent test function and* *the time derivative on the test function. However, since later we need higher regularity to prove Fréchet differentiability of $\tau \mapsto \widetilde{v}(\tau)$, we use the setting* (2.19).

If τ enjoys higher regularity than we want to assume here, e.g., $\tau \in W^{2,\infty}(\Omega_{\text{ref}})$, then the standard $W(I;V(\Omega))$ framework can be used.

Next, we apply the transformation rule for integrals and the identities

$$(\nabla_{\widetilde{x}}\widetilde{v}) \circ_{\widetilde{x}} \tau = (\tau')^{-T}\nabla v, \quad (\mathrm{div}_{\widetilde{x}}\, \widetilde{v}) \circ_{\widetilde{x}} \tau = \mathrm{tr}(\tau'^{-T}\nabla v)$$

to obtain a variational formulation on the domain Ω_{ref} that is equivalent to (2.20). To handle the source term, we assume for all $\Omega \in \mathcal{O}_{\text{ad}}$ that

$$\widetilde{f} \in L^1(I;\mathbf{L}^\mu(\Omega)) \quad \text{with} \quad \mu > 1 \;\text{ if } d = 2, \quad \mu = 6/5 \;\text{ if } d = 3.$$

We set $\mu' = \mu/(\mu - 1)$. We obtain the

Navier-Stokes equations transformed to the reference domain. Find $(v, p) \in W(I;\mathbf{H}^1_0(\Omega_{\text{ref}})) \times L^2(I;L^2_0(\Omega_{\text{ref}}))$ with $v_t \in L^2(I;\mathbf{L}^2(\Omega_{\text{ref}}))$ such that

$$\begin{aligned}
&\langle (w, q, w_0), E((v(t), p(t)), \tau)\rangle_{(\mathbf{H}^1_0\times L^2_0\times\mathbf{L}^2)(\Omega_{\text{ref}}),(\mathbf{H}^1_0\times L^2_0\times\mathbf{L}^2)(\Omega_{\text{ref}})^*}\\
&:= (v_t(t), wg_1(\tau))_{\mathbf{L}^2(\Omega_{\text{ref}})} + \nu(\nabla v(t), g_2(\tau)\nabla w)_{\mathbf{L}^2(\Omega_{\text{ref}})} + \widehat{b}(v(t), v(t), w, g_3(\tau))\\
&\quad - (p(t), \mathrm{tr}(g_3(\tau)\nabla w))_{L^2_0(\Omega_{\text{ref}})} - \langle \widetilde{f}(t) \circ_{\widetilde{x}} \tau, w - g_1(\tau)\rangle_{\mathbf{L}^\mu(\Omega_{\text{ref}}),\mathbf{L}^{\mu'}(\Omega_{\text{ref}})}\\
&\quad + (q, \mathrm{tr}(g_3(\tau)\nabla v(t)))_{L^2_0(\Omega_{\text{ref}})} + (v(\cdot, 0) - \widetilde{v}_0(\tau(\cdot)), w_0)_{\mathbf{L}^2(\Omega_{\text{ref}})}\\
&= 0, \qquad\qquad \forall (w, q, w_0) \in (\mathbf{H}^1_0 \times L^2_0 \times \mathbf{L}^2)(\Omega_{\text{ref}}) \;\text{ for a.a. } t \in I.
\end{aligned} \tag{2.21}$$

For $\tau = \mathrm{id}$ we recover directly the weak formulation (2.20) on the domain $\Omega = \Omega_{\text{ref}}$, for general $\tau \in T_{\text{ad}}$ we obtain an equivalent form of (2.20) on the domain $\Omega = \tau(\Omega_{\text{ref}})$. This follows from the fact that for $\tau \in T_{\text{ad}}$ the map $\widetilde{v} \mapsto \widetilde{v} \circ \tau$ is a homeomorphism between the spaces $\mathbf{L}^r(\Omega)$ and $\mathbf{L}^r(\Omega_{\text{ref}})$, $\mathbf{W}^{1,r}(\Omega)$ and $\mathbf{W}^{1,r}(\Omega_{\text{ref}})$, as well as $\mathbf{W}^{1,r}_0(\Omega)$ and $\mathbf{W}^{1,r}_0(\Omega_{\text{ref}})$, respectively, for all $r \geq 1$, see [33, Lemma 4.1]. Moreover, the pressures $p(t) \in L^2_0(\Omega_{\text{ref}})$ and $\widetilde{p}(t) \in L^2_0(\tau(\Omega_{\text{ref}}))$ are related via the homeomorphism

$$\tilde{p}(t) \in L^2_0(\tau(\Omega_{\text{ref}})) \mapsto p(t) := P_0(\Omega_{\text{ref}})\tilde{p}(t, \tau(\cdot)) \in L^2_0(\Omega_{\text{ref}})$$

with the projection $P_0(\Omega_{\text{ref}})$ according to (2.6). Note that not necessarily $\tilde{p}(t, \tau(\cdot)) \in L^2_0(\Omega_{\text{ref}})$, but $p(t) = P_0(\Omega_{\text{ref}})\tilde{p}(t, \tau(\cdot))$ is its corresponding representative in $L^2_0(\Omega_{\text{ref}})$ and we have $\|\tilde{p}(t, \tau(\cdot))\|_{L^2(\Omega_{\text{ref}})/\mathbb{R}} = \|p(t)\|_{L^2_0(\Omega_{\text{ref}})}$.

3. Differentiability theorem. The following Theorem 3.1 provides a framework to show Fréchet differentiability of $u \in U \mapsto y(u) \in Y$ at $u_0 \in U$, where $y(u) \in Y^+ \hookrightarrow Y$ denotes the unique solution of $E(y, u) = 0$. Here, we take care of the fact that for nonlinear problems often $E_y(y(u_0), u_0) \in L(Y^+, Z)$ only admits an inverse $E_y(y(u_0), u_0)^{-1} \in L(Z_0, Y)$ for a subspace $Z_0 \subsetneq Z$. For the application to shape differentiability of the Navier-Stokes equations we will choose Y as a sum of a space with solenoidal velocity fields with corresponding pressure and an image space of Bogovskiĭ's operator. While for the first space we can apply standard theory for the linearized Navier-Stokes equations, the second space deals with the inhomogenity in the divergence equation.

The basic framework is motivated by typical norm gap differentiability results for nonlinear problems, see e.g. [45, Thm. 2.1], but incorporates different regularity requirements for the directions and the point of linearization in the definition of the Fréchet derivative. This procedure turns out to be very useful, since the higher regularity is usually only needed in the point of linearization, such that much weaker continuity requirements on the solution operator are neccessary.

THEOREM 3.1. *Let $u_0 \in U$ and let Z, Y^+, Y_i be Banach spaces (possibly depending on the fixed u_0) with $Y^+ \hookrightarrow Y_i$ for $i = 1, 2$. Let $E : Y^+ \times U \to Z$ be a given operator and $y_0 \in Y^+$ such that $E(y_0, u_0) = 0$.*

Moreover, we assume the existence of a unique locally bounded solution map $S : \mathcal{U} \to Y^+$ in a neighbourhood $\mathcal{U} \subset U$ of u_0, i.e., $E(S(u), u) = 0$ for all $u \in \mathcal{U}$ and $\|S(u)\|_{Y^+} \le C_y$ uniformly in $u \in \mathcal{U}$ for a constant $C_y > 0$.

Furthermore, with functions $K_i : \mathbb{R}^+ \to \mathbb{R}^+$, $i = 1, 2$, $K_i(x) \to 0$ as $x \in \mathbb{R}^+ \to 0$, let E satisfy the following assumptions.

1. *$E(y, \cdot) : U \to Z$ is Fréchet differentiable at $u_0 \in \mathcal{U}$ uniformly (i.e., the remainder term can be estimated uniformly) for all $y \in Y^+$ satisfying $\|y\|_{Y^+} \le C_y$.*
2. *There exists an operator $E_y : Y^+ \times U \to L(Y^+, Z)$ such that for all $u \in \mathcal{U} \subset U$ and all $y_0, y \in Y^+$ with $\|y_0\|_{Y^+}, \|y\|_{Y^+} \le C_y$, it holds*
$$\|E(y, u) - E(y_0, u) - E_y(y_0, u)[y - y_0]\|_Z \le CK_1(\|y - y_0\|_{Y_2})\|y - y_0\|_{Y_1}$$
with a constant $C > 0$ independent of y_0, y, u.
3. *For all $e \in Y^+$ with $\|e\|_{Y^+} \le 2C_y$ it holds for $\|h\|_U \to 0$ that*
$$\|(E_y(y_0, u_0 + h) - E_y(y_0, u_0))\, e\|_Z \le C\big(\|h\|_U \|e\|_{Y_2} + K_2(\|h\|_U)\|e\|_{Y_1}\big).$$
4. *For $i = 1, 2$ the solution operator S is continuous at u_0 in Y_i, i.e., it holds $\|S(u_0 + h) - S(u_0)\|_{Y_i} \to 0$ as $\|h\|_U \to 0$.*

Finally, let there exist Banach spaces Y with $Y^+ \hookrightarrow Y \hookrightarrow Y_1$ and $Z_0 \subset Z$ equipped with the norm of Z (Y, Z_0 may again depend on the fixed u_0) such that

5. *The partial derivative $E_y(y_0, u_0) \in L(Y^+, Z)$ from 2. admits an inverse $E_y(y_0, u_0)^{-1} \in L(Z_0, Y)$. Moreover, $E_u(y_0, u_0) \in L(U, Z_0)$ and $E_y(y_0, u_0)(y - y_0) \in Z_0$ for all $y \in Y^+$.*

Then $S : \mathcal{U} \to Y$ is Fréchet differentiable at u_0 and the derivative is given by
$$S'(u_0)h = -E_y(y_0, u_0)^{-1} E_u(y_0, u_0)h \tag{3.1}$$
for all $h \in U$.

Proof. For all $h \in U$ we define $S'(u_0)h$ via (3.1) and in the following we will show that $S'(u_0)$ is indeed the Fréchet derivative of $S : \mathcal{U} \to Y$ at u_0. To this end, we define $(\cdot) := (y_0, u_0)$ with $y_0 = S(u_0)$ and $y = S(u_0 + h)$. Using 5., we have
$$S(u_0 + h) - S(u_0) = y - y_0 = E_y(\cdot)^{-1} E_y(\cdot)[y - y_0]$$
and thus by the definition of $S'(u_0)h$
$$\begin{aligned}
&\|S(u_0 + h) - S(u_0) - S'(u_0)h\|_Y \\
&\le \left\|E_y(y_0, u_0)^{-1} \left(E_y(y_0, u_0)[y - y_0] + E_u(y_0, u_0)[h]\right)\right\|_Y \\
&\le C\|E_y(y_0, u_0)[y - y_0] + E_u(y_0, u_0)[h]\|_Z,
\end{aligned}$$
where we have used assumption 5. and that Z_0 is equipped with the Z-norm. After inserting some terms adding up to 0, the previous estimate yields
$$\begin{aligned}
&\|S(u_0 + h) - S(u_0) - S'(u_0)h\|_Y \\
&\quad \le C\Big(\|E(y_0, u_0 + h) - E(\cdot) - E_u(\cdot)[h]\|_Z \\
&\qquad + \|E(S(u_0 + h), u_0 + h) - E(y_0, u_0 + h) - E_y(y_0, u_0 + h)[S(u_0 + h) - S(u_0)]\|_Z \\
&\qquad + \|\left(E_y(y_0, u_0 + h) - E_y(\cdot)\right)[S(u_0 + h) - S(u_0)]\|_Z\Big).
\end{aligned}$$

Now the assumptions 1., 2. and 3. on E yield

$$\begin{aligned}
&\|S(u_0+h)-S(u_0)-S'(u_0)h\|_Y \\
&\quad \le o(\|h\|_U) + C\Big(\|S(u_0+h)-S(u_0)\|_{Y_1} K_1\left(\|S(u_0+h)-S(u_0)\|_{Y_2}\right) \\
&\qquad + \|h\|_U \|S(u_0+h)-S(u_0)\|_{Y_2} + K_2(\|h\|_U)\|S(u_0+h)-S(u_0)\|_{Y_1}\Big).
\end{aligned} \tag{3.2}$$

Since $E_u(y_0,u_0) \in L(U,Z_0)$ by 5. and

$$\|S(u_0+h)-S(u_0)\|_{Y_1} \le \|S(u_0+h)-S(u_0)-S'(u_0)h\|_{Y_1} + \left\|E_y(\cdot)^{-1}E_u(\cdot)[h]\right\|_{Y_1},$$

we obtain by assumption 5. and $Y \hookrightarrow Y_1$

$$\|S(u_0+h)-S(u_0)\|_{Y_1} \le C\|S(u_0+h)-S(u_0)-S'(u_0)h\|_Y + C\|h\|_U$$

with a constant $C>0$. By using (3.2) and

$$\|S(u_0+h)-S(u_0)\|_{Y_2}, K_1\left(\|S(u_0+h)-S(u_0)\|_{Y_2}\right), K_2(\|h\|_U) \to 0,$$

as $\|h\|_U \to 0$, this implies

$$\|S(u_0+h)-S(u_0)-S'(u_0)h\|_Y \le o(\|S(u_0+h)-S(u_0)-S'(u_0)h\|_Y) + o(\|h\|_U),$$

which immediately results in

$$\|S(u_0+h)-S(u_0)-S'(u_0)h\|_Y = o(\|h\|_U),$$

as $\|h\|_U \to 0$. ☐

The following lemma shows that the continuity assumptions on S in 4. are automatically satisfied if the assumptions on E are slightly extended.

LEMMA 3.2 (Continuity of the solution operator). *Let assumptions 1.-3. of Theorem 3.1 be satisfied and assume that moreover there holds*

6. *The mapping $y \in Y^+ \mapsto E_y(y,u_0) \in L(Y^+,Z)$ is continuous and the averaged partial derivative*

$$A(y,y_0) := \int_0^1 E_y(y_0+t(y-y_0),u_0)\,dt \in L(Y^+,Z)$$

admits an inverse $A(y,y_0)^{-1} \in L(Z_0,Y)$ that is uniformly bounded for all $y \in Y^+$ with $\|y\|_{Y^+} \le C_y$. Moreover, $A(y,y_0)[y-y_0] \in Z_0$ for all $y \in Y^+$ and $E_u(y_0,u_0) \in L(U,Z_0)$.

If in addition

$$\|S(u_0+h)-S(u_0)\|_{Y_1} \to 0 \Longrightarrow \|S(u_0+h)-S(u_0)\|_{Y_2} \to 0 \tag{3.3}$$

as $\|h\|_U \to 0$, then the solution operator $S : \mathcal{U} \to Y^+$ is continous in Y_k, i.e.

$$\|S(u_0+h)-S(u_0)\|_{Y_k} \to 0$$

for $k=1,2$ as $\|h\| \to 0$.

Proof. For all $h \in U$ with $\|h\|_U$ small enough, we have $u_0+h \in \mathcal{U}$ and for abbreviation we again define $y := S(u_0+h)$ and $y_0 := S(u_0)$. Since $E(y,u_0+h) = E(y_0,u_0) = 0$, we have

$$E(y,u_0)-E(y_0,u_0) = -\Big(E(y,u_0+h)-E(y,u_0)-E_u(y,u_0)[h]+E_u(y,u_0)[h]\Big)$$

and the uniform Fréchet differentiability with respect to u by 1. thus gives

$$\|E(y,u_0)\|_Z = \|E(y,u_0) - E(y_0,u_0)\|_Z \le C\|h\|_U \tag{3.4}$$

for sufficiently small $\|h\|_U > 0$.

By assumptions 2. and 6. the mapping $t \in [0,t] \mapsto E(y_0 + t(y-y_0), u_0) \in Z$ is continuously differentiable with derivative $E_y(y_0 + t(y-y_0), u_0)[y-y_0]$. Hence, by the definition of $A(y,y_0)$ we have with 6.

$$A(y,y_0)[y-y_0] = E(y,u_0) - E(y_0,u_0) = E(y,u_0) \in Z_0,$$

and thus again by 6.

$$S(u_0+h) - S(u_0) = y - y_0 = A(y,y_0)^{-1}A(y,y_0)[y-y_0] = A(y,y_0)^{-1}E(y,u_0).$$

Now, $A(y,y_0)^{-1} \in L(Z_0,Y)$ and $Y \hookrightarrow Y_1$ yield immediately

$$\|S(u_0+h) - S(u_0)\|_{Y_1} \le C\|E(y,u_0)\|_Z \le C\|h\|_U.$$

Using (3.3), this implies $\|S(u_0+h) - S(u_0)\|_{Y_k} \to 0$, as $\|h\|_U \to 0$ for $k = 1,2$. □

4. Application to the Navier-Stokes shape optimization problem.

4.1. Shape differentiability of the solution operator. Since solenoidality is not preserved under transformations, it is clear that the velocity $v(\tau)$, given as the solution of (2.21) for $\tau \neq \mathrm{id}$, is not necessarily divergence free. If $S(\tau) := (v(\tau), p(\tau))$ denotes the full solution of (2.21), it turns out that the non-solenoidality of $v(\tau)$ also transfers to $\frac{d}{d\tau}v(\tau)_{|\tau=\mathrm{id}}$. Thus, in order to obtain $\frac{d}{d\tau}S(\tau)_{|\tau=\mathrm{id}}$ one has to solve a linearized Navier-Stokes type equation with non zero right hand side in the divergence equation. To show the existence of a bounded inverse of this linearized Navier-Stokes operator, we Bogovskiĭ's operator B to deal with the inhomogeneous right hand side of the divergence equation and can then solve a standard linearized Navier-Stokes equation in the space of solenoidal velocity fields.

In the following we want to apply Theorem 3.1 and Lemma 3.2 to show Fréchet differentiability of $\tau \mapsto (v(\tau), p(\tau))$ at $\tau = \mathrm{id}$ in a suitable Banach space setting. To this end, we have to choose suitable Banach spaces Y^+, Y and Y_i, $i = 1,2$ for the solutions, Z, Z_0 for the image spaces of the differential operator and U for the control variables. A possible choice of Banach spaces for the solution and image spaces, which we will use in the following, is given in Table 4.1.

Here, all function spaces are defined on Ω_{ref} and for Banach spaces X_1, X_2 that are embedded in a common Banach space, the sum $X_1 + X_2$ and the intersection $X_1 \cap X_2$ are Banach spaces with the usual norms

$$\|z\|_{X_1+X_2} := \inf_{z=x+y, x\in X_1, y\in X_2} \|x\|_{X_1} + \|y\|_{X_2},$$
$$\|z\|_{X_1\cap X_2} := \max\big(\|z\|_{X_1}, \|z\|_{X_2}\big).$$

Moreover, we use the abbreviation $P(I;X_1,X_2) := L^2(I;X_1) \cap W^{1,1}(I;X_2)$ with its dual $P(I;X_1,X_2)^* = L^2(I;X_1^*) + (W^{1,1}(I;X_2))^*$ for reflexive Banach spaces X_1, X_2 such that $X_1 \cap X_2$ is dense in X_1, X_2. We note that Z_0 is well defined, since for $(z_1, z_{\mathrm{div}}, z_0) \in Z$ we have $z_{\mathrm{div}} \in C([0,T];(H^{\frac{3}{2}-\varepsilon})^*)$.

Finally, we choose as the space for the domain variations

$$U := (\mathbf{W}^{1,\infty} \cap \mathbf{W}^{1+s,r})(\Omega_{\mathrm{ref}})$$

	v	p
Y	$W(I;V)+W(I;\mathbf{H}_0^1)$	$P(I;L_0^2,\mathrm{cl}_{H_0^{-1}}(L_0^2))^*$
Y_1	$L^\infty(I;\mathbf{L}^2)\cap L^2(I;\mathbf{H}_0^1)$	$P(I;L_0^2,\mathrm{cl}_{H_0^{-1}}(L_0^2))^*$
Y_2	$L^\infty(I;\mathbf{L}^3)\cap L^2(I;\mathbf{H}_0^1)\cap H^1(I;\mathbf{H}_0^{1-\varepsilon})$	$L^2(I;L_0^2)$
Y^+	$H^1(I;\mathbf{H}_0^1)$	$L^2(I;L_0^2)$
Z	$\big(L^2(I;V)+P(I;\mathbf{H}_0^1,\mathbf{L}^2)\big)^*\times\big(L^2(I;L_0^2)\cap H^1(I;(H^{\frac{3}{2}-\varepsilon})^*)\big)\times\mathbf{L}^2$	
Z_0	$\{(z_1,z_{\mathrm{div}},z_0)\in Z:\ z_{\mathrm{div}}(0)=\mathrm{div}(z_0)\}$	$\|\cdot\|_{Z_0}=\|\cdot\|_Z$

TABLE 4.1
Possible choice of spaces for the application of Theorem 3.1

and equip it with its natural norm

$$\|\cdot\|_U := \|\cdot\|_{\mathbf{W}^{1,\infty}(\Omega_{\mathrm{ref}})} + \|\cdot\|_{\mathbf{W}^{1+s,r}(\Omega_{\mathrm{ref}})}$$

for arbitrary but fixed $1<r<2$ and $0<s<\frac{1}{r}$ from Assumption 2.3.

For the rest of the paper, ε from the definition of Y_2 is strictly connected to s,r via

$$\varepsilon = \frac{3sr}{2(sr+3)} \in (0,\frac{3}{8}). \tag{4.1}$$

We start with the following observation.

LEMMA 4.1. *Let Assumption 2.3 hold. Then for any $\widehat{\tau}\in T_{\mathrm{ad}}$ there exists $\rho>0$, such that $\tau\in T_{\mathrm{ad}}$ holds for all $\tau\in U$ with $\|\tau-\widehat{\tau}\|_U<\rho$.*

Proof. Let $\widehat{\tau}\in T_{\mathrm{ad}}$ and $\tau\in U$ with $\|\tau-\widehat{\tau}\|_U<\rho$. Since $T_{\mathrm{ad}}\subset U$ is by Assumption 2.3 relatively open, it remains to show that for $\rho>0$ small enough $\tau(\Omega_{\mathrm{ref}})$ is a bounded Lipschitz domain, $\tau^{-1}\in(\mathbf{W}^{1,\infty}\cap\mathbf{W}^{1+s,r})(\tau(\Omega_{\mathrm{ref}}))$ and $g_1(\tau)>\delta$ a.e. on Ω_{ref}.

$\widehat{\Omega}:=\widehat{\tau}(\Omega_{\mathrm{ref}})$ is a bounded Lipschitz domain by the definition of T_{ad}. By [5, Lem. 3] there exists $\rho'>0$ such that $\|h^\tau\|_{\mathbf{W}^{1,\infty}(\mathbb{R}^d)}<\rho'$ implies that $(\mathrm{id}+h^\tau)(\widehat{\Omega})$ is a bounded Lipschitz domain. The same holds after reducing $\rho'>0$ also for $\|h^\tau\|_{\mathbf{W}^{1,\infty}(\widehat{\Omega})}<\rho'$, since there exists a linear bounded extension operator $\mathbf{W}^{1,\infty}(\widehat{\Omega})\to\mathbf{W}^{1,\infty}(\mathbb{R}^d)$ for the bounded Lipschitz domain $\widehat{\Omega}$, see e.g. [41, Thm. 5, p. 181].

We have $\tau(\Omega_{\mathrm{ref}})=(\mathrm{id}+(\tau-\widehat{\tau})\circ\widehat{\tau}^{-1})(\widehat{\Omega})$ and $h^\tau:=(\tau-\widehat{\tau})\circ\widehat{\tau}^{-1}$ satisfies $\|h^\tau\|_{\mathbf{W}^{1,\infty}(\widehat{\Omega})}\le(1+\|\widehat{\tau}^{-1}\|_{\mathbf{W}^{1,\infty}(\widehat{\Omega})})\rho$. For $\rho>0$ small enough we have $\|h^\tau\|_{\mathbf{W}^{1,\infty}(\widehat{\Omega})}<\rho'$ and thus $\tau(\Omega_{\mathrm{ref}})$ is a bounded Lipschitz domain.

Since $\operatorname{essinf}_{x\in\Omega_{\mathrm{ref}}} g_1(\widehat{\tau})\ge\delta'>\delta>0$, continuity yields $\operatorname{essinf}_{x\in\Omega_{\mathrm{ref}}} g_1(\tau)>\delta$ after a possible reduction of $\rho>0$. Moreover, we have

$$\|\tau(x)-\tau(y)\|\ge\|\widehat{\tau}(x)-\widehat{\tau}(y)\|-2\|\tau-\widehat{\tau}\|_{\mathbf{W}^{1,\infty}(\Omega_{\mathrm{ref}})}\|x-y\|\ge(\|\widehat{\tau}^{-1}\|^{-1}_{\mathbf{W}^{1,\infty}(\widehat{\Omega})}-2\rho)\|x-y\|.$$

Hence, possible after reducing $\rho>0$ the inverse mapping τ^{-1} exists and satisfies $\tau^{-1}\in\mathbf{W}^{1,\infty}(\tau(\Omega_{\mathrm{ref}}))$.

Finally, we show that $(\tau^{-1})'\in\mathbf{W}^{s,r}(\tau(\Omega_{\mathrm{ref}}))$. Since $(\tau^{-1})'(\tau(x))=\tau'(x)^{-1}$ and

$$\begin{aligned}\|(\tau^{-1})'(\tau(x))-(\tau^{-1})'(\tau(y))\| &= \|\tau'(x)^{-1}(\tau'(y)-\tau'(x))\tau'(y)^{-1}\|\\ &\le \|\tau^{-1}\|^2_{\mathbf{W}^{1,\infty}(\tau(\Omega_{\mathrm{ref}}))}\|\tau'(x)-\tau'(y)\|\end{aligned}$$

we obtain by using $\|x-y\| \le \|\tau^{-1}\|_{\mathbf{W}^{1,\infty}(\tau(\Omega_{\mathrm{ref}}))}\|\tau(x)-\tau(y)\|$

$$\begin{aligned}
&\int_{\tau(\Omega_{\mathrm{ref}})^2} \frac{\|(\tau^{-1})'(x)-(\tau^{-1})'(y)\|^r}{\|x-y\|^{d+sr}}\,dx\,dy \\
&\quad\le C\|\tau^{-1}\|^{2r}_{\mathbf{W}^{1,\infty}(\tau(\Omega_{\mathrm{ref}}))}\int_{\Omega_{\mathrm{ref}}{}^2} \frac{\|\tau'(x)-\tau'(y)\|^r}{\|\tau(x)-\tau(y)\|^{d+sr}} g_1(\tau)\,dx\,dy \\
&\quad\le C\|\tau^{-1}\|^{2r+d+sr}_{\mathbf{W}^{1,\infty}(\tau(\Omega_{\mathrm{ref}}))}\int_{\Omega_{\mathrm{ref}}{}^2} \frac{\|\tau'(x)-\tau'(y)\|^r}{\|x-y\|^{d+sr}}\,dx\,dy \\
&\quad\le C\|\tau^{-1}\|^{2r+d+sr}_{\mathbf{W}^{1,\infty}(\tau(\Omega_{\mathrm{ref}}))}\|\tau\|^r_{\mathbf{W}^{1+s,r}(\Omega_{\mathrm{ref}})}.
\end{aligned}$$

□

REMARK 4.2.

1. *Since in the current section we will only work with the operator formulation* (2.21) *on the reference domain* Ω_{ref}*, we will omit* Ω_{ref} *in scalar products, norms and integrals.*
2. *Instead of updating constants in every step of an estimate, we will simply use a generic constant* $C>0$ *and write* $C(A_1, A_2, \dots)$ *whenever the dependence on* $A_1, A_2, \dots$ *is relevant.*

With the help of the previously defined spaces we are now able to proof the following main theorem of this paper.

THEOREM 4.3. *Let* $d \in \{2,3\}$ *and let Assumption 2.3 hold true. Then the following holds.*

i): There exists a neighborhood $\mathcal{U} \subset U$ *of* id *such that the mapping*

$$\tau \in \mathcal{U} \subset U \mapsto (v(\tau), p(\tau)) \in Y \hookrightarrow \left(L^2(I;\mathbf{H}_0^1(\Omega_{\mathrm{ref}})) \cap C(\overline{I};\mathbf{L}^2(\Omega_{\mathrm{ref}}))\right) \times \left(L^2(I;L_0^2(\Omega_{\mathrm{ref}})) + \left(W^{1,1}\big(I;\mathrm{cl}_{H_0^{-1}}(L_0^2(\Omega_{\mathrm{ref}}))\big)\right)^*\right)$$

is well defined and Fréchet differentiable at $\tau = \mathrm{id}$*. The derivative is given by* (3.1)*, i.e., with the linearized Navier-Stokes operator* $E_{(v,p)}$ *in* (4.4) *and* E_τ *in* (4.3) *one has*

$$(v'(\mathrm{id}), p'(\mathrm{id})) = -E_{(v,p)}(v(\mathrm{id}), p(\mathrm{id}), \mathrm{id})^{-1}E_\tau(v(\mathrm{id}), p(\mathrm{id}), \mathrm{id}).$$

ii): For any $\widehat{\tau} \in T_{\mathrm{ad}}$ *there exists a neighborhood* $\mathcal{U} \subset U$ *of* $\widehat{\tau}$ *such that the mapping*

$$\tau \in \mathcal{U} \subset U \mapsto (v(\tau), p(\tau)) \in (L^2(I;\mathbf{H}_0^1(\Omega_{\mathrm{ref}})) \cap C(\overline{I};\mathbf{L}^2(\Omega_{\mathrm{ref}}))) \times \left(L^2(I;L_0^2(\Omega_{\mathrm{ref}})) + \left(W^{1,1}\big(I;\mathrm{cl}_{H_0^{-1}}(L_0^2(\Omega_{\mathrm{ref}}))\big)\right)^*\right)$$

is well defined and Fréchet differentiable at $\tau = \widehat{\tau}$*.*

In i) and ii) the remainder estimate on $\mathcal{U}$ *depends only on* $\sup_{\tau\in\mathcal{U}}\|(v(\tau), p(\tau))\|_{Y^+}$*.*

Proof. For i) we apply Theorem 3.1 and Lemma 3.2. The assumptions will be verified for the spaces in Table 4.1 in Lemmas 4.7–4.15.

For ii) we apply i) on $\widehat{\Omega} := \widehat{\tau}(\Omega_{\mathrm{ref}})$ instead of Ω_{ref}. Since $\widehat{\Omega}$ is a bounded Lipschitz domain by the definition of T_{ad}, we can apply i) with $\widehat{\Omega}$ instead of Ω_{ref} as reference domain. Hence, let $U(\widehat{\Omega}) = (\mathbf{W}^{1,\infty} \cap \mathbf{W}^{1+s,r})(\widehat{\Omega})$, $\mathcal{U}(\widehat{\Omega}) \subset U(\widehat{\Omega})$ be a sufficiently small neighborhood of id and denote by $\widetilde{\tau} \in \mathcal{U}(\widehat{\Omega}) \mapsto (v(\widetilde{\tau};\widehat{\Omega}), p(\widetilde{\tau};\widehat{\Omega})) \in (L^2(I;\mathbf{H}_0^1(\widehat{\Omega})) \cap C(\overline{I};\mathbf{L}^2(\widehat{\Omega}))) \times P(I;L_0^2(\widehat{\Omega}), \mathrm{cl}_{H_0^{-1}}(L_0^2(\widehat{\Omega})))^*$ the mapping in i) with $\widehat{\Omega}$ instead of Ω_{ref} as reference domain, which is Fréchet differentiable at $\widetilde{\tau} = \mathrm{id}$. By the following Lemma 4.5, we find a neighborhood $\mathcal{U} \subset U$ of $\widehat{\tau}$ such that $\tau \circ \widehat{\tau}^{-1} \in \mathcal{U}(\widehat{\Omega})$ for all $\tau \in \mathcal{U}$ and

$$(v(\tau), p(\tau)) = (v(\tau \circ \widehat{\tau}^{-1}; \widehat{\Omega}) \circ_x \widehat{\tau}, P_0(\Omega_{\mathrm{ref}})(p(\tau \circ \widehat{\tau}^{-1}; \widehat{\Omega}) \circ_x \widehat{\tau}))$$

with the projection $P_0(\Omega_{\mathrm{ref}})$ in (2.6), since both, $(v(\tau), p(\tau)) \circ_x \tau^{-1}$ as well as $(v(\tau \circ \widehat{\tau}^{-1}; \widehat{\Omega}), p(\tau \circ \widehat{\tau}^{-1}; \widehat{\Omega})) \circ_x (\widehat{\tau} \circ \tau^{-1})$, are the same solution of the Navier-Stokes equations on $\tau(\Omega_{\mathrm{ref}})$ modulo a constant in the pressure. Furthermore, the mapping $\tau \in U \mapsto \tau \circ \widehat{\tau}^{-1} \in U(\widehat{\Omega})$ is in $L(U, U(\widehat{\Omega}))$ and

$$v \in L^2(I; \mathbf{H}_0^1(\widehat{\Omega})) \cap C(\overline{I}; \mathbf{L}^2(\widehat{\Omega})) \mapsto v \circ_x \widehat{\tau} \in L^2(I; \mathbf{H}_0^1(\Omega_{\mathrm{ref}})) \cap C(\overline{I}; \mathbf{L}^2(\Omega_{\mathrm{ref}}))$$

is obviously linear and bounded. Finally, the mapping

$$\begin{aligned}
&p \in P(I; L_0^2(\widehat{\Omega}), \mathrm{cl}_{H_0^{-1}}(L_0^2(\widehat{\Omega})))^* \mapsto Ap \in P(I; L_0^2(\Omega_{\mathrm{ref}}), \mathrm{cl}_{H_0^{-1}}(L_0^2(\Omega_{\mathrm{ref}})))^*,\\
&\langle Ap, w\rangle_{P(I;L_0^2(\Omega_{\mathrm{ref}}),\mathrm{cl}_{H_0^{-1}}(L_0^2(\Omega_{\mathrm{ref}})))^*, P(I;L_0^2(\Omega_{\mathrm{ref}}),\mathrm{cl}_{H_0^{-1}}(L_0^2(\Omega_{\mathrm{ref}})))}\\
&\quad := \lim_{k\to\infty} \langle p, g_1(\widehat{\tau}^{-1}) w_k \circ_x \widehat{\tau}^{-1}\rangle_{P(I;L_0^2(\widehat{\Omega}),\mathrm{cl}_{H_0^{-1}}(L_0^2(\widehat{\Omega})))^*, P(I;L_0^2(\widehat{\Omega}),\mathrm{cl}_{H_0^{-1}}(L_0^2(\widehat{\Omega})))},
\end{aligned}$$

where $(w_k)_{k\in\mathbb{N}} \in W^{1,1}(I; L_0^2(\Omega_{\mathrm{ref}}))$ denotes an arbitrary sequence satisfying

$$\|w_k - w\|_{P(I;L_0^2(\Omega_{\mathrm{ref}}),\mathrm{cl}_{H_0^{-1}}(L_0^2(\Omega_{\mathrm{ref}})))} \to 0 \quad \text{as } k \to \infty,$$

is also linear and bounded and by construction $p(\tau \circ \widehat{\tau}^{-1}; \widehat{\Omega}) \in L^2(I; L_0^2(\widehat{\Omega}))$ yields

$$Ap(\tau \circ \widehat{\tau}^{-1}; \widehat{\Omega}) = P_0(\Omega_{\mathrm{ref}})(p(\tau \circ \widehat{\tau}^{-1}; \widehat{\Omega}) \circ_x \widehat{\tau}) = p(\tau).$$

Thus, the concatenation of Fréchet differentiable and linear continous mappings yields the Fréchet differentiability of $\tau \mapsto (v(\tau), p(\tau))$ at $\tau = \widehat{\tau}$ in the correct spaces. $\square$

Remark 4.4. *For technical reasons in the proof of Lemma 4.9 we restrict ourselves to prove Theorem 4.3 for $0 < s < \frac{1}{r}$ with $1 < r < 2$, since this immediately yields Theorem 4.3 for arbitrary $s > 0$, $r > 1$.*

Lemma 4.5. *Let Assumption 2.3 hold and let $\widehat{\tau} \in T_{\mathrm{ad}}$. Then there exists a constant $C > 0$ such that for all $v \in (\mathbf{W}^{1,\infty} \cap \mathbf{W}^{1+s,r})(\Omega_{\mathrm{ref}})$*

$$\|v \circ \widehat{\tau}^{-1}\|_{(\mathbf{W}^{1,\infty}\cap\mathbf{W}^{1+s,r})(\widehat{\tau}(\Omega_{\mathrm{ref}}))} \le C\|v\|_{(\mathbf{W}^{1,\infty}\cap\mathbf{W}^{1+s,r})(\Omega_{\mathrm{ref}})}.$$

Proof. By the definition of T_{ad} the set $\widehat{\Omega} := \widehat{\tau}(\Omega_{\mathrm{ref}})$ is a bounded Lipschitz domain and $\widehat{\tau} \in (\mathbf{W}^{1,\infty} \cap \mathbf{W}^{1+s,r})(\Omega_{\mathrm{ref}}), \widehat{\tau}^{-1} \in (\mathbf{W}^{1,\infty} \cap \mathbf{W}^{1+s,r})(\widehat{\Omega})$. Therefore it is easy to see that there exists $C > 0$ such that for all $v \in (\mathbf{W}^{1,\infty} \cap \mathbf{W}^{1+s,r})(\Omega_{\mathrm{ref}})$

$$\|v \circ \widehat{\tau}^{-1}\|_{(\mathbf{W}^{1,\infty}\cap\mathbf{W}^{1,r})(\widehat{\tau}(\Omega_{\mathrm{ref}}))} \le C\|v\|_{(\mathbf{W}^{1,\infty}\cap\mathbf{W}^{1,r})(\Omega_{\mathrm{ref}})}.$$

We have $\nabla(v \circ \widehat{\tau}^{-1})^T = v' \circ \widehat{\tau}^{-1} \cdot (\widehat{\tau}^{-1})'$ and therefore

$$\begin{aligned}
\|\nabla(v \circ \widehat{\tau}^{-1})(x) - \nabla(v \circ \widehat{\tau}^{-1})(y)\| \le{}& \|\widehat{\tau}^{-1}\|_{\mathbf{W}^{1,\infty}(\widehat{\Omega})} \|v' \circ \widehat{\tau}^{-1}(x) - v' \circ \widehat{\tau}^{-1}(y)\|\\
&+ \|v\|_{\mathbf{W}^{1,\infty}(\Omega_{\mathrm{ref}})} \|(\widehat{\tau}^{-1})'(x) - (\widehat{\tau}^{-1})'(y)\|.
\end{aligned}$$

Hence,

$$\int_{\widehat{\Omega}^2} \frac{\|\nabla(v\circ\widehat{\tau}^{-1})(x)-\nabla(v\circ\widehat{\tau}^{-1})(y)\|^r}{\|x-y\|^{d+sr}}\,dx\,dy \le C\Big(\|v\|^r_{\mathbf{W}^{1,\infty}(\Omega_{\mathrm{ref}})}\|\widehat{\tau}^{-1}\|^r_{\mathbf{W}^{1+s,r}(\Omega_{\mathrm{ref}})}$$
$$+\|\widehat{\tau}^{-1}\|^r_{\mathbf{W}^{1,\infty}(\widehat{\Omega})}\int_{{\Omega_{\mathrm{ref}}}^2} g_1(\widehat{\tau})\frac{\|v'(x)-v'(y)\|^r}{\|\widehat{\tau}(x)-\widehat{\tau}(y)\|^{d+sr}}\,dx\,dy\Big)$$
$$\le C\left(\|v\|^r_{\mathbf{W}^{1,\infty}(\Omega_{\mathrm{ref}})}\|\widehat{\tau}^{-1}\|^r_{\mathbf{W}^{1+s,r}(\Omega_{\mathrm{ref}})}+\|\widehat{\tau}^{-1}\|^{r+d+sr}_{\mathbf{W}^{1,\infty}(\widehat{\Omega})}\|v\|^r_{\mathbf{W}^{1+s,r}(\Omega_{\mathrm{ref}})}\right),$$

where we have used that

$$\|x-y\|=\|\widehat{\tau}^{-1}(\widehat{\tau}(x))-\widehat{\tau}^{-1}(\widehat{\tau}(y))\|\le\|\widehat{\tau}^{-1}\|_{\mathbf{W}^{1,\infty}(\widehat{\Omega})}\|\widehat{\tau}(x)-\widehat{\tau}(y)\|.$$

□

4.2. Uniform boundedness of the state variables. We start by showing that $v(\tau)$ and $p(\tau)$ are uniformly bounded in a neighborhood of $\tau=\mathrm{id}$ with respect to $\|\cdot\|_{Y^+}$.

LEMMA 4.6. *Let $d\in\{2,3\}$ and let Assumption 2.3 hold. Let (v,p) be the solution of (2.21) for $\tau\in T_{\mathrm{ad}}$. Then there exist $\rho, C_y>0$ with*

$$\begin{aligned}&\|\tau-\mathrm{id}\|_{\mathbf{W}^{1,\infty}(\Omega_{\mathrm{ref}})}<\rho \quad\Longrightarrow\\ &\|v\|_{C(\bar{I};\mathbf{H}^1_0)}+\|v_t\|_{L^2(I;\mathbf{H}^1_0)}+\|v_t\|_{L^\infty(I;\mathbf{L}^2)}+\|p\|_{L^2(I;L^2_0)}\le C_y,\end{aligned}\tag{4.2}$$

i.e. it holds $\|(v(\tau),p(\tau))\|_{Y^+}\le C_y$ uniformly for all $\tau\in T_{\mathrm{ad}}$ with $\|\tau-\mathrm{id}\|_U\le\rho$.

Proof. The assertion for the velocity can be verified by analyzing the proofs of the regularity of the states (see e.g. [43]) because the bounds of the norms depend only on the data $\widetilde{f},\widetilde{v}_0,\nu$ and the dimension d. From this also the bound for the pressure can be obtained as in the proof of Lemma 2.2 (even the boundedness of $\|p\|_{L^\infty(I;L^2_0)}$ can be shown) and by using the fact that the inverse of the gradient operator is uniformly bounded with respect to small $\mathbf{W}^{1,\infty}$-perturbations of Lipschitz domains. □

4.3. Differentiability of E with respect to domain variations. In the following we will prove assumption 1 of Theorem 3.1, i.e. the Fréchet differentiability of $E((v,p),\cdot):U\to Z$ at $\tau=\mathrm{id}$.

LEMMA 4.7. *Let $d\in\{2,3\}$ and let Assumption 2.3 hold. Then there exists a linearized operator $E_\tau(\cdot,\mathrm{id}):Y^+\to L(U,Z)$, such that*

$$\|E((v,p),\tau)-E((v,p),\mathrm{id})-E_\tau((v,p),\mathrm{id})[\tau-\mathrm{id}]\|_Z=o(\|\tau-\mathrm{id}\|_{\mathbf{W}^{1,\infty}(\Omega_{\mathrm{ref}})})$$

holds true for $\|\tau-\mathrm{id}\|_{\mathbf{W}^{1,\infty}(\Omega_{\mathrm{ref}})}\to 0$ uniformly for all $(v,p)\in Y^+$ with $\|(v,p)\|_{Y^+}\le C_y$.

Proof. Formal linearization yields for $E_\tau((v,p),\mathrm{id})[h^\tau]$ with arbitrary $h^\tau\in U$ and $(v,p)\in Y^+$ the candidat

$$\begin{aligned}&\langle(w,q,w_0),E_\tau((v(t),p(t)),\mathrm{id})[h^\tau]\rangle_{\mathbf{H}^1_0\times L^2_0\times\mathbf{L}^2,(\mathbf{H}^1_0\times L^2_0\times\mathbf{L}^2)^*}\\ &:=(v_t(t),g_1'(\mathrm{id})[h^\tau]w)_{\mathbf{L}^2}+\nu(\nabla v(t),g_2'(\mathrm{id})[h^\tau]\nabla w)_{\mathbf{L}^2}+\widehat{b}(v(t),v(t),w,g_3'(\mathrm{id})[h^\tau])\\ &\quad+(p(t),\mathrm{tr}(g_3'(\mathrm{id})[h^\tau]\nabla w))_{L^2_0}-\Big(\tfrac{d}{d\tau}(\widetilde{f}(t)\circ_x\tau\det\tau')|_{\tau=\mathrm{id}}[h^\tau],w\Big)_{\mathbf{H}^{-1},\mathbf{H}^1_0}\\ &\quad+(q,\mathrm{tr}(g_3'(\mathrm{id})[h^\tau]\nabla v(t)))_{L^2_0}-\big(\tfrac{d}{d\tau}(\widetilde{v}_0(\tau(\cdot)))|_{\tau=\mathrm{id}}[h^\tau],w_0\big)_{\mathbf{L}^2},\end{aligned}\tag{4.3}$$

for all $w \in \mathbf{H}_0^1(\Omega_{\text{ref}}), q \in L_0^2(\Omega_{\text{ref}}), w_0 \in \mathbf{L}^2(\Omega_{\text{ref}})$ and for a.a. $t \in I$. Thus, with $S(t) := (v(t), p(t))$ and $r_i(\tau, \text{id}) := g_i(\tau) - g_i(\text{id}) - g_i'(\text{id})[\tau - \text{id}]$ for $i = 1, 2, 3$, the remainder is given by

$$
\begin{aligned}
&\big\langle (w, q, w_0), E(S(t), \tau) - E(S(t), \text{id}) - E_\tau(S(t), \text{id})[\tau - \text{id}] \big\rangle_{\mathbf{H}_0^1 \times L_0^2 \times \mathbf{L}^2, (\mathbf{H}_0^1 \times L_0^2 \times \mathbf{L}^2)^*} \\
&\quad = (v_t(t), r_1(\tau, \text{id}) w)_{\mathbf{L}^2} + \nu (\nabla v(t), r_2(\tau, \text{id}) \nabla w)_{\mathbf{L}^2} + \widehat{b}(v(t), v(t), w, r_3(\tau, \text{id})) \\
&\qquad + (p(t), \text{tr}(r_3(\tau, \text{id}) \nabla w))_{L_0^2} + (q, \text{tr}(r_3(\tau, \text{id}) \nabla v(t)))_{L_0^2} \\
&\qquad - \Big(\widetilde{f}(t) \circ_x \tau \det \tau' - \widetilde{f}(t) - \tfrac{d}{d\tau} (\widetilde{f}(t) \circ_x \tau \det \tau')|_{\tau = \text{id}} (\tau - \text{id}), w \Big)_{\mathbf{H}^{-1}, \mathbf{H}_0^1} \\
&\qquad - \big(\widetilde{v}_0(\tau(\cdot)) - \widetilde{v}_0 - \tfrac{d}{d\tau} (\widetilde{v}_0(\tau(\cdot)))|_{\tau = \text{id}} (\tau - \text{id}), w_0 \big)_{\mathbf{L}^2}.
\end{aligned}
$$

for all $w \in \mathbf{H}_0^1(\Omega_{\text{ref}}), q \in L_0^2(\Omega_{\text{ref}}), w_0 \in \mathbf{L}^2(\Omega_{\text{ref}})$ and for a.a. $t \in I$.

Since $L^2(I; \mathbf{H}^{-1}) \hookrightarrow (L^2(I; V) + P(I; \mathbf{H}_0^1, \mathbf{L}^2))^*$, it is sufficient to estimate each term involved in the left hand side of the momentum equation in $L^2(I; \mathbf{H}^{-1})$. Using $\|r_i(\tau, \text{id})\|_{L^\infty} = o(\|\tau - \text{id}\|_{\mathbf{W}^{1,\infty}(\Omega_{\text{ref}})})$ for $i = 1, 2, 3$ and $\mathbf{L}^2 \hookrightarrow \mathbf{H}^{-1}$, we obtain

$$
\begin{aligned}
\|v_t r_1(\tau, \text{id})\|_{L^2(I; \mathbf{L}^2)} &\leq \|v_t\|_{L^2(I; \mathbf{L}^2)} o(\|\tau - \text{id}\|_{\mathbf{W}^{1,\infty}(\Omega_{\text{ref}})}), \\
\|\nu (\nabla v_i, r_2(\tau, \text{id}) \nabla(\cdot))\|_{L^2(I; \mathbf{H}^{-1})} &\leq \nu \|v\|_{L^2(I; \mathbf{H}_0^1)} o(\|\tau - \text{id}\|_{\mathbf{W}^{1,\infty}(\Omega_{\text{ref}})}), \\
\|\widehat{b}_I(v, v, \cdot, r_3(\tau, \text{id}))\|_{L^2(I; \mathbf{H}^{-1})} &\leq \|v\|_{L^\infty(I; \mathbf{H}_0^1)} \|v\|_{L^2(I; \mathbf{H}_0^1)} o(\|\tau - \text{id}\|_{\mathbf{W}^{1,\infty}(\Omega_{\text{ref}})}), \\
\|(p, \text{tr}(r_3(\tau, \text{id}) \nabla \cdot))\|_{L^2(I; \mathbf{H}^{-1})} &\leq \|p\|_{L^2(I; L_0^2)} o(\|\tau - \text{id}\|_{\mathbf{W}^{1,\infty}(\Omega_{\text{ref}})}),
\end{aligned}
$$

where we have made use of Hölder's inequality and Lemma 2.6. Because of the regularity of $\widetilde{f}$ in Assumption 2.3 we have

$$
\Big\| \widetilde{f} \circ_x \tau \det \tau' - \widetilde{f} - \tfrac{d}{d\tau} (\widetilde{f} \circ_x \tau \det \tau')|_{\tau = \text{id}} (\tau - \text{id}) \Big\|_{L^2(I; \mathbf{H}^{-1})} = o(\|\tau - \text{id}\|_{\mathbf{W}^{1,\infty}(\Omega_{\text{ref}})}),
$$

which overall shows the differentiability of the terms involved in the momentum equation in the correct space.

Concerning the term arising from the divergence equation it holds

$$
\begin{aligned}
\|\text{tr}(r_3(\tau, \text{id}) \nabla v)\|_{L^2(I; L_0^2)} &\leq \|v\|_{L^2(I; \mathbf{H}_0^1)} o(\|\tau - \text{id}\|_{\mathbf{W}^{1,\infty}(\Omega_{\text{ref}})}) \\
\|\text{tr}(r_3(\tau, \text{id}) \nabla v_t)\|_{L^2(I; (H^{3/2-\varepsilon})^*)} &\leq C \|\text{tr}(r_3(\tau, \text{id}) \nabla v_t)\|_{L^2(I; L^2)} \\
&\leq \|v_t\|_{L^2(I; \mathbf{H}_0^1)} o(\|\tau - \text{id}\|_{\mathbf{W}^{1,\infty}(\Omega_{\text{ref}})}).
\end{aligned}
$$

Due to the regularity assumptions of $\widetilde{v}_0$ we have

$$
\|\widetilde{v}_0(\tau(\cdot)) - \widetilde{v}_0 - \tfrac{d}{d\tau} (\widetilde{v}_0(\tau(\cdot)))|_{\tau = \text{id}} (\tau - \text{id})\|_{\mathbf{L}^2} = o(\|\tau - \text{id}\|_{\mathbf{W}^{1,\infty}(\Omega_{\text{ref}})}),
$$

which finally proves

$$
\|E((v, p), \tau) - E((v, p), \text{id}) - E_\tau((v, p), \text{id})[\tau - \text{id}]\|_Z = o(\|\tau - \text{id}\|_{\mathbf{W}^{1,\infty}(\Omega_{\text{ref}})})
$$

for $\|\tau - \text{id}\|_{\mathbf{W}^{1,\infty}(\Omega_{\text{ref}})} \to 0$ uniformly for all $(v, p) \in Y^+$ with $\|(v, p)\|_{Y^+} \leq C_y$.

By using exactly the same estimates as in the above estimate of the remainder, but with $o(\|\tau - \text{id}\|_{\mathbf{W}^{1,\infty}(\Omega_{\text{ref}})})$ replaced by $O(\|h^\tau\|_{\mathbf{W}^{1,\infty}(\Omega_{\text{ref}})})$, we obtain also the continuity result $\|E_\tau((\bar{v}, \bar{p}), \text{id}) h^\tau\|_Z = O(\|h^\tau\|_{\mathbf{W}^{1,\infty}(\Omega_{\text{ref}})})$. $\square$

4.4. Differentiability property of E with respect to the state variables. Next we prove that the linearization of E with respect to the state variables satisfies the remainder estimate of assumption 2 of Theorem 3.1.

LEMMA 4.8. *For $d \in \{2,3\}$ there exists a linearized operator $E_y : Y^+ \times U \to L(Y^+, Z)$ such that for all $\tau \in U$ with $\|\tau - \mathrm{id}\|_{\mathbf{W}^{1,\infty}(\Omega_{\mathrm{ref}})}$ sufficiently small and all $\bar{y} := (\bar{v}, \bar{p}), y := (v, p) \in Y^+$ with $\|\bar{y}\|_{Y^+}, \|y\|_{Y^+} \le C_y$ it holds*

$$\|E(y,\tau) - E(\bar{y},\tau) - E_y(\bar{y},\tau)[y-\bar{y}]\|_Z \le C\|y-\bar{y}\|_{Y_2}\|y-\bar{y}\|_{Y_1},$$

with a constant $C > 0$ independent of $y, \bar{y}, \tau$.

Proof. Clearly, for an arbitrary direction $h := (h^v, h^p) \in Y^+$ the candidat for the linearization $E_y(\bar{y},\tau)[h]$ is given by

$$\begin{aligned}
&\langle (w,q,w_0), E_y((\bar{v}(t),\bar{p}(t)),\tau)[h]\rangle_{\mathbf{H}_0^1\times L_0^2\times\mathbf{L}^2,(\mathbf{H}_0^1\times L_0^2\times\mathbf{L}^2)^*}\\
&\quad := ((h^v(t))_t, g_1(\tau)w)_{\mathbf{L}^2} + \nu(\nabla h^v(t), g_2(\tau)\nabla w)_{\mathbf{L}^2} + \widehat{b}(h^v(t),\bar{v}(t),w,g_3(\tau))\\
&\qquad + \widehat{b}(\bar{v}(t),h^v(t),w,g_3(\tau)) + (h^p(t),\mathrm{tr}(g_3(\tau)\nabla w))_{L_0^2}\\
&\qquad + (q,\mathrm{tr}(g_3(\tau)\nabla h^v(t)))_{L_0^2} + (w_0, h^v(\cdot,0))_{\mathbf{L}^2}
\end{aligned} \tag{4.4}$$

for all $w \in \mathbf{H}_0^1(\Omega_{\mathrm{ref}}), q \in L_0^2(\Omega_{\mathrm{ref}}), w_0 \in \mathbf{L}^2(\Omega_{\mathrm{ref}})$ and for a.a. $t \in I$.

The boundedness of the linearized operator $E_y(\bar{y},\tau)$ from Y^+ to Z is easily checked, such that we only show the remainder estimate. Since E is linear with respect to the state variables in every term except the nonlinearity $\widehat{b}$, the only non-vanishing term in the remainder is given by

$$\begin{aligned}
&\widehat{b}_I(v,v,\cdot,g_3(\tau)) - \widehat{b}_I(\bar{v},\bar{v},\cdot,g_3(\tau)) - \widehat{b}_I(v-\bar{v},\bar{v},\cdot,g_3(\tau)) - \widehat{b}_I(\bar{v},v-\bar{v},\cdot,g_3(\tau))\\
&\quad = \widehat{b}_I(v-\bar{v},v-\bar{v},\cdot,g_3(\tau)).
\end{aligned}$$

By utilizing Hölder's inequality with $\frac{5}{6} = \frac{1}{3} + \frac{1}{2}$ and $\mathbf{L}^{\frac{6}{5}} \hookrightarrow \mathbf{H}^{-1}$ we obtain

$$\begin{aligned}
\|\widehat{b}_I(v-\bar{v},v-\bar{v},\cdot,g_3(\tau))\|_{L^2(I;\mathbf{H}^{-1})} &\le C\|g_3(\tau)\|_{\mathbf{L}^\infty}\|v-\bar{v}\|_{L^\infty(I;\mathbf{L}^3)}\|\nabla(v-\bar{v})\|_{L^2(I;\mathbf{L}^2)}\\
&\le C\|v-\bar{v}\|_{L^\infty(I;\mathbf{L}^3)}\|v-\bar{v}\|_{L^2(I;\mathbf{H}_0^1)},
\end{aligned}$$

which concludes the proof, since $L^2(I;\mathbf{H}^{-1}) \hookrightarrow (L^2(I;V) + P(I;\mathbf{H}_0^1,\mathbf{L}^2))^*$. $\square$

4.5. Lipschitz estimate of E_y. In this subsection, we prove the Lipschitz type estimate of E_y stated in assumption 3 of Theorem 3.1.

LEMMA 4.9. *Let $d \in \{2,3\}$ and let Assumption 2.3 hold. Denote by $\bar{y} := (\bar{v}, \bar{p})$ the solutions of (2.21) for $\tau = \mathrm{id}$. Then for all $e = (e^v, e^p) \in Y^+$ with $\|e\|_{Y^+} \le 2C_y$ the linearized operator $E_y(\bar{y},\mathrm{id}) \in L(Y^+, Z)$ from (4.4) satisfies*

$$\|(E_y(\bar{y},\tau) - E_y(\bar{y},\mathrm{id}))[e]\|_Z \le C\|\tau - \mathrm{id}\|_{(\mathbf{W}^{1,\infty}\cap\mathbf{W}^{1+s,r})(\Omega_{\mathrm{ref}})}\|e\|_{Y_2},$$

with a constant $C > 0$, provided $\|\tau - \mathrm{id}\|_{(\mathbf{W}^{1,\infty}\cap\mathbf{W}^{1+s,r})(\Omega_{\mathrm{ref}})}$ is sufficiently small.

Proof. By (4.4), $(E_y(\bar{y},\tau) - E_y(\bar{y},\mathrm{id}))[e]$ is given by

$$\begin{aligned}
&\langle (w,q,w_0), (E_y(\bar{y}(t),\tau) - E_y(\bar{y}(t),\mathrm{id}))[e(t)]\rangle_{\mathbf{H}_0^1\times L_0^2\times\mathbf{L}^2,(\mathbf{H}_0^1\times L_0^2\times\mathbf{L}^2)^*}\\
&\quad = ((e^v(t))_t, (g_1(\tau)-g_1(\mathrm{id}))w)_{\mathbf{L}^2} + \nu(\nabla e^v(t), (g_2(\tau)-g_2(\mathrm{id}))\nabla w)_{\mathbf{L}^2}\\
&\qquad + \widehat{b}(e^v(t),\bar{v}(t),w,(g_3(\tau)-g_3(\mathrm{id}))) + \widehat{b}(\bar{v}(t),e^v(t),w,(g_3(\tau)-g_3(\mathrm{id})))\\
&\qquad + (e^p(t),\mathrm{tr}((g_3(\tau)-g_3(\mathrm{id}))\nabla w))_{L_0^2} + (q,\mathrm{tr}((g_3(\tau)-g_3(\mathrm{id}))\nabla e^v(t)))_{L_0^2}
\end{aligned} \tag{4.5}$$

for all $w \in \mathbf{H}_0^1(\Omega_{\text{ref}}), q \in L_0^2(\Omega_{\text{ref}}), w_0 \in \mathbf{L}^2(\Omega_{\text{ref}})$ and for a.a. $t \in I$.

Since $L^2(I;\mathbf{H}^{-1}) \hookrightarrow (L^2(I;V) + P(I;\mathbf{H}_0^1,\mathbf{L}^2))^*$, it is again sufficient to estimate the terms of the momentum equation in $L^2(I;\mathbf{H}^{-1})$. By using $\|g_i(\tau) - g_i(\text{id})\|_{\mathbf{L}^\infty} = O(\|\tau - \text{id}\|_{\mathbf{W}^{1,\infty}(\Omega_{\text{ref}})})$ for $i = 1,2,3$ and $\mathbf{L}^2 \hookrightarrow \mathbf{H}^{-1}$, we have for the first two terms on the right hand side of (4.5)

$$
\begin{aligned}
&\|(e_t^v, (g_1(\tau) - g_1(\text{id}))\cdot)\|_{L^2(I;\mathbf{H}^{-1})} + \|\nu(\nabla e^v, (g_2(\tau) - g_2(\text{id}))\nabla(\cdot))\|_{L^2(I;\mathbf{H}^{-1})} \\
&\quad \le C \left(\left\|(g_1(\tau) - g_1(\text{id}))^T e_t^v\right\|_{L^2(I;\mathbf{L}^2)} + \left\|\nu(g_2(\tau) - g_2(\text{id}))^T \nabla e^v\right\|_{L^2(I;\mathbf{L}^2)} \right) \\
&\quad \le C\|\tau - \text{id}\|_{\mathbf{W}^{1,\infty}(\Omega_{\text{ref}})} \left(\|e_t^v\|_{L^2(I;\mathbf{H}_0^{1-\varepsilon})} + \|e^v\|_{L^2(I;\mathbf{H}_0^1)} \right),
\end{aligned}
$$

where $\varepsilon \in (0, \frac{3}{8})$ is given by (4.1). Furthermore, by using $\mathbf{H}_0^1 \hookrightarrow \mathbf{L}^6$, $\mathbf{L}^{\frac{6}{5}} \hookrightarrow \mathbf{H}^{-1}$ and Hölder's inequality with $\frac{5}{6} = \frac{1}{3} + \frac{1}{2} + \frac{1}{\infty}$ the trilinear terms satisfy

$$
\begin{aligned}
&\left\|\widehat{b}_I(e^v, \bar{v}, \cdot, (g_3(\tau) - g_3(\text{id}))) + \widehat{b}_I(\bar{v}, e^v, \cdot, (g_3(\tau) - g_3(\text{id})))\right\|_{L^2(I;\mathbf{H}^{-1})} \\
&\quad \le C \left(\|\nabla \bar{v}\|_{L^\infty(I;\mathbf{L}^2)} \|e^v\|_{L^2(I;\mathbf{L}^3)} + \|\bar{v}\|_{L^\infty(I;\mathbf{L}^3)} \|\nabla e^v\|_{L^2(I;\mathbf{L}^2)} \right) \|g_3(\tau) - g_3(\text{id})\|_{\mathbf{L}^\infty} \\
&\quad \le C\|\bar{v}\|_{L^\infty(I;\mathbf{H}_0^1)} \|e^v\|_{L^2(I;\mathbf{H}^1)} \|\tau - \text{id}\|_{\mathbf{W}^{1,\infty}(\Omega_{\text{ref}})}.
\end{aligned}
$$

The remaining term arising from the momentum equation can be estimated via

$$
\|(e^p, \text{tr}((g_3(\tau) - g_3(\text{id}))\nabla\cdot))\|_{L^2(I;\mathbf{H}^{-1})} \le C\|\tau - \text{id}\|_{\mathbf{W}^{1,\infty}(\Omega_{\text{ref}})} \|e^p\|_{L^2(I;L_0^2)}.
$$

In order to conclude the proof, we have to estimate the term arising from the divergence equation. To this end, we start with the obvious estimate

$$
\|\text{tr}((g_3(\tau) - g_3(\text{id}))\nabla e^v)\|_{L^2(I;L_0^2)} \le C\|\tau - \text{id}\|_{\mathbf{W}^{1,\infty}(\Omega_{\text{ref}})} \|e^v\|_{L^2(I;\mathbf{H}_0^1)},
$$

such that it remains to consider $\|\text{tr}((g_3(\tau) - g_3(\text{id}))\nabla e_t^v)\|_{L^2(I;(H^{3/2-\varepsilon})^*)}$.

Fix $i \in \{1,\dots,d\}$ and let $a^T := (e^i)^T(g_3(\tau) - g_3(\text{id}))$, with e^i denoting the i-th unit vector. If e_i^v denotes the i-th component of e^v we thus have to estimate $\|a^T \nabla (e_i^v)_t\|_{L^2(I;(H^{3/2-\varepsilon})^*)}$.

The divergence operator $\text{div} \in L(\mathbf{H}^1(\Omega_{\text{ref}}), L^2(\Omega_{\text{ref}}))$ can be extended to the bounded operator $\text{div} \in L(\mathbf{L}^2(\Omega_{\text{ref}}), H^{-1}(\Omega_{\text{ref}}))$ via

$$
\langle \text{div} v, w\rangle_{H^{-1}(\Omega_{\text{ref}}),H_0^1(\Omega_{\text{ref}})} = -(v, \nabla w)_{\mathbf{L}^2(\Omega_{\text{ref}}),\mathbf{L}^2(\Omega_{\text{ref}})} \quad \forall\, w \in H_0^1(\Omega_{\text{ref}}).
$$

Using the characterization of Bessel potential spaces via complex interpolation, cf. [18, Thm 3.1], one obtains

$$
\mathbf{H}^\varepsilon = [\mathbf{L}^2, \mathbf{H}^1]_\varepsilon, \quad H_0^{1-\varepsilon} = [L^2, H_0^1]_{1-\varepsilon}
$$

and [44, Section 1.11.3 and 1.9.3] yield

$$
H^{-1+\varepsilon} = \left(H_0^{1-\varepsilon}\right)^* = [L^2, H^{-1}]_{1-\varepsilon} = [H^{-1}, L^2]_\varepsilon.
$$

Hence, complex interpolation gives

$$
\text{div} \in L\left([\mathbf{L}^2, \mathbf{H}^1]_\varepsilon, [H^{-1}, L^2]_\varepsilon\right) = L\left(\mathbf{H}^\varepsilon, H^{-1+\varepsilon}\right).
$$

Since e^v and thus e_t^v vanish on the boundary, and $\nabla e_t^v \in L^2(I;\mathbf{L}^2)$ we obtain

$$
\begin{aligned}
\langle w, a^T\nabla(e_i^v)_t\rangle_{H^{\frac{3}{2}-\varepsilon},(H^{\frac{3}{2}-\varepsilon})^*} &= (wa, \nabla(e_i^v)_t)_{\mathbf{L}^2} \leq \|\operatorname{div}(wa)\|_{H^{-1+\varepsilon}}\|(e_i^v)_t\|_{H_0^{1-\varepsilon}} \\
&\leq C\|wa\|_{\mathbf{H}^\varepsilon}\|(e_i^v)_t\|_{H_0^{1-\varepsilon}} \qquad \forall w \in H^{3/2-\varepsilon}(\Omega_{\text{ref}}).
\end{aligned}
\tag{4.6}
$$

Since Ω_{ref} is a bounded Lipschitz domain, the extension by zero operator E_0 : $W^{\alpha,q}(\Omega_{\text{ref}}) \to W^{\alpha,q}(\mathbb{R}^d)$ is bounded for $0 \leq \alpha < \frac{1}{q}$, $1 < q < \infty$, see e.g. [1, Section 14.5]. Furthermore, since on the bounded Lipschitz domain Ω_{ref} the Bessel potential spaces $H^\alpha(\Omega_{\text{ref}})$ and the Sobolev Slobodeckij spaces $W^{\alpha,2}(\Omega_{\text{ref}})$ are equivalent for all $\alpha > 0$, cf. [44, Thm. 2.3.2 (d) and Rem. 4.4.2.2], and also $W^{\alpha,2}(\mathbb{R}^d) = H^\alpha(\mathbb{R}^d)$, the definition of the Sobolev Slobodeckij norm immediately yields

$$
\|wa\|_{\mathbf{H}^\varepsilon(\Omega_{\text{ref}})} \leq C\|wa\|_{\mathbf{W}^{\varepsilon,2}(\Omega_{\text{ref}})} \leq C\|E_0wE_0a\|_{\mathbf{W}^{\varepsilon,2}(\mathbb{R}^d)} \leq C\|E_0wE_0a\|_{\mathbf{H}^\varepsilon(\mathbb{R}^d)}. \tag{4.7}
$$

Now let $1 < r < 2$ and $0 < s < \frac{1}{r}$ as in Assumption 2.3 and define $\varepsilon \in (0, \frac{3}{8})$ according to (4.1). Then $\varepsilon = \theta s \in (0, \frac{3}{8})$ with $\theta := \frac{\frac{3}{2}r}{sr+3} \in (\frac{3}{8}, 1)$. By the Sobolev embedding theorem we have for $d = 2, 3$

$$
H^{\frac{3}{2}-\varepsilon}(\Omega_{\text{ref}}) \hookrightarrow L^\eta(\Omega_{\text{ref}}) \quad \text{with } \eta := \frac{3}{\varepsilon}.
$$

Since $\frac{1}{\eta} + \frac{\theta}{r} = \frac{1}{2}$, the Runst-Sickel Lemma [9, Lem. 6] yields

$$
\begin{aligned}
&\|E_0wE_0a\|_{\mathbf{H}^\varepsilon(\mathbb{R}^d)} \\
&\quad\leq C(\|E_0a\|_{\mathbf{L}^\infty(\mathbb{R}^d)}\|E_0w\|_{H^\varepsilon(\mathbb{R}^d)} + \|E_0w\|_{L^\eta(\mathbb{R}^d)}\|E_0a\|^\theta_{\mathbf{W}^{s,r}(\mathbb{R}^d)}\|E_0a\|^{1-\theta}_{\mathbf{L}^\infty(\mathbb{R}^d)}) \\
&\quad\leq C\|w\|_{H^{3/2-\varepsilon}(\Omega_{\text{ref}})}(\|a\|_{\mathbf{L}^\infty(\Omega_{\text{ref}})} + \|a\|_{\mathbf{W}^{s,r}(\Omega_{\text{ref}})}),
\end{aligned}
$$

where we have used $\varepsilon < \frac{1}{2}$ and thus $\|E_0w\|_{H^\varepsilon(\mathbb{R}^d)} \leq C\|w\|_{H^\varepsilon(\Omega_{\text{ref}})} \leq C\|w\|_{H^{\frac{3}{2}-\varepsilon}(\Omega_{\text{ref}})}$. Combining this with (4.6) and (4.7) we have shown that

$$
\|a^T\nabla(e_i^v)_t\|_{L^2(I;(H^{3/2-\varepsilon})^*)} \leq C(\|a\|_{\mathbf{L}^\infty} + \|a\|_{\mathbf{W}^{s,r}})\|(e_i^v)_t\|_{L^2(I;H_0^{1-\varepsilon})}.
$$

Finally, there exists $\rho > 0$ such that $\|a(x)-a(y)\| = \|g_3(\tau)(x)-g_3(\tau)(y)\| \leq L\|\tau'(x)-\tau'(y)\|$ on $\|\tau - \text{id}\|_{\mathbf{W}^{1,\infty}(\Omega_{\text{ref}})} < \rho$ with Lipschitz constant L for all $x, y \in \Omega_{\text{ref}}$. Hence,

$$
\begin{aligned}
\|a\|^r_{\mathbf{W}^{s,r}} &= \|g_3(\tau) - g_3(\text{id})\|^r_{\mathbf{W}^{s,r}} \\
&= \|g_3(\tau) - g_3(\text{id})\|^r_{\mathbf{L}^r} + \int_{\Omega^2_{\text{ref}}} \frac{\|g_3(\tau(x)) - g_3(\tau(y))\|^r}{\|x-y\|^{d+sr}}\,dx\,dy \\
&\leq C\|\tau - \text{id}\|^r_{\mathbf{W}^{1,r}} + L^r\int_{\Omega^2_{\text{ref}}} \frac{\|\tau'(x) - \tau'(y))\|^r}{\|x-y\|^{d+sr}}\,dx\,dy \leq C\|\tau - \text{id}\|^r_{\mathbf{W}^{1+s,r}},
\end{aligned}
$$

and with $\|a\|_{\mathbf{L}^\infty} \leq \|g_3(\text{id}) - g_3(\tau)\|_{\mathbf{L}^\infty} \leq C\|\tau - \text{id}\|_{\mathbf{W}^{1,\infty}}$ as $\|\tau - \text{id}\|_{\mathbf{W}^{1,\infty}(\Omega_{\text{ref}})} \to 0$ we arrive at

$$
\begin{aligned}
&\|\operatorname{tr}((g_3(\tau) - g_3(\text{id}))\nabla e_t^v)\|_{L^2(I;(H^{3/2-\varepsilon})^*)} \\
&\qquad\leq C\|\tau - \text{id}\|_{(\mathbf{W}^{1,\infty}\cap\mathbf{W}^{1+s,r})(\Omega_{\text{ref}})}\|e_t^v\|_{L^2(I;\mathbf{H}_0^{1-\varepsilon})},
\end{aligned}
$$

which concludes the proof. □

4.6. Bogovskiĭ's operator. In order to verify assumption 5 of Theorem 3.1 or more generally assumption 6 of Lemma 3.2, we have to show that $E_y(\bar{y}, \mathrm{id}) \in L(Y^+, Z)$ admits a bounded inverse $E_y(\bar{y}, \mathrm{id})^{-1} \in L(Z_0, Y)$ and that the averaged derivative $A(y, \bar{y}) \in L(Y^+, Z)$ admits a bounded inverse $A(y, \bar{y})^{-1} \in L(Z_0, Y)$, respectively. Thus, we have to invert a linearized Navier-Stokes equation with nonhomogeneous right hand side in the divergence equation.

In order to deal with low domain and data regularities, we use Bogovskiĭ's operator to obtain a particular solution of the divergence equation and obtain then a standard linearized Navier-Stokes equations for solenoidal velocity fields that can be handled by standard techniques. Bogovskiĭ's operator is an antiderivative of the divergence operator and admits the following useful stability estimates that have been shown by Geissert, Heck and Hieber [16].

LEMMA 4.10. *Let $d \in \{2,3\}$ and $\Omega \subset \mathbb{R}^d$ be a bounded Lipschitz domain. Then there exists a continuous linear operator $B : L^2_0(\Omega) \to H^1_0(\Omega)^d$ such that $\mathrm{div}(Bf) = f$ for all $f \in L^2_0(\Omega)$. Moreover, B can be extended continuously to a bounded operator $B : H^\alpha_0(\Omega) \to H^{\alpha+1}_0(\Omega)^d$ for $-3/2 < \alpha \le 0$ such that*

$$\mathrm{div}(Bf) = f \quad \forall\, f \in H^\alpha_0(\Omega) \text{ with } \langle f, 1\rangle_{H^\alpha_0(\Omega), H^{-\alpha}(\Omega)} = 0.$$

Here, for $\alpha > 0$ the space $H^{-\alpha}_0(\Omega)$ is defined by $H^{-\alpha}_0(\Omega) = H^\alpha(\Omega)^*$.

Proof. The proof is given in [16]. □

To deal now with the divergence equation in the construction of $E_y(\bar{y}, \mathrm{id})^{-1} \in L(Z_0, Y)$ or more generally of $A(y, \bar{y})^{-1} \in L(Z_0, Y)$ we will use the operator

$$D : Z \to Y, \quad D(z_1, z_{\mathrm{div}}, z_0) := (B z_{\mathrm{div}}, 0)\,, \tag{4.8}$$

where B denotes the Bogovskiĭ operator of Lemma 4.10.

LEMMA 4.11. *Let $d \in \{2,3\}$ and let Assumption 2.3 hold. Then the operator D in* (4.8) *is bounded, i.e. $D \in L(Z, Y)$, and for all $z = (z_1, z_{\mathrm{div}}, z_0) \in Z$ the image $(v, p) = Dz \in Y$ satisfies* $\mathrm{div}\, v = z_{\mathrm{div}}$.

Proof. Recall that for $z = (z_1, z_{\mathrm{div}}, z_0) \in Z$ we have $Dz = (Bz_{\mathrm{div}}, 0)$. Then Lemma 4.10 immediately yields

$$\|Bz_{\mathrm{div}}\|_{L^2(I;\mathbf{H}^1_0)} \le C\|z_{\mathrm{div}}\|_{L^2(I;L^2_0)}$$

and due to $(H^{\frac{1}{2}-\varepsilon})^* \hookrightarrow (H^1)^* \hookrightarrow H^{-1}$, we also have

$$\|(Bz_{\mathrm{div}})_t\|_{L^2(I;\mathbf{H}^{-1})} \le C\|B(z_{\mathrm{div}})_t\|_{L^2(I;(\mathbf{H}^{\frac{1}{2}-\varepsilon})^*)} \le C\|(z_{\mathrm{div}})_t\|_{L^2(I;(H^{\frac{3}{2}-\varepsilon})^*)},$$

which proves $D \in L(Z, W(I; \mathbf{H}^1_0) \times \{0\})$ and thus $D \in L(Z, Y)$. Finally $\mathrm{div}\, Bz_{\mathrm{div}} = z_{\mathrm{div}}$ follows from Lemma 4.10. □

4.7. Invertibility of E_y. In the following Lemmas 4.12 and 4.14 we show assumption 6 of Lemma 3.2, which also directly implies assumption 5 of Theorem 3.1.

LEMMA 4.12. *Let $d \in \{2,3\}$ and let Assumption 2.3 hold. Let $\bar{y} := (\bar{v}, \bar{p})$ be the solution of* (2.21) *for $\tau = \mathrm{id}$ and let $E_y(\cdot, \mathrm{id}) : Y^+ \to L(Y^+, Z)$ denote the partial derivative from* (4.4). *Then $y \in Y^+ \mapsto E_y(y, \mathrm{id}) \in L(Y^+, Z)$ is continuous and the averaged partial derivative*

$$A(y, \bar{y}) := \int_0^1 E_y(\bar{y} + t(y - \bar{y}), \mathrm{id})\, dt \in L(Y^+, Z)$$

admits an inverse $A(y,\bar{y})^{-1} \in L(Z_0, Y)$ *that is uniformly bounded for all* $y \in Y^+$ *with* $\|y\|_{Y^+} \le C_y$.

To prove this lemma, we have to consider an equation of the form

$$A(y,\bar{y})(h^v, h^p) = (z_1, z_{\mathrm{div}}, z_0)$$

with $z = (z_1, z_{\mathrm{div}}, z_0) \in Z_0$. We now make the ansatz

$$(h^v, h^p) = Dz + (h_0^v, h^p) = (Bz_{\mathrm{div}}, 0) + (h_0^v, h^p),$$

which leads to a standard linearized Navier-Stokes equation

$$A(y,\bar{y})(h_0^v, h^p) = (\mathrm{id} - A(y,\bar{y})D)z \in Z_{0,h} := \{(\tilde{z}_1, 0, \tilde{z}_0) \in Z_0 :\ \tilde{z}_0 \in H\}$$

that can be handled by standard techniques. In other words, we show that $A(y,\bar{y})^{-1} \in L(Z_0, Y)$ can be obtained by

$$A(y,\bar{y})^{-1} = D + A(y,\bar{y})^{-1}(\mathrm{id} - A(y,\bar{y})D),$$

where $A(y,\bar{y})^{-1}$ on the right hand side is the standard solution operator for the linearized Navier-Stokes equation with solenoidal velocity field.

To prepare the proof of Lemma 4.12, we start with the following auxiliary result.

LEMMA 4.13. *Let the assumptions of Lemma 4.12 hold and define the Banach space*

$$Z_{0,h} := \{(z_1, 0, z_0) \in Z_0 :\ z_0 \in H\}, \quad \|\cdot\|_{Z_{0,h}} := \|\cdot\|_{Z_0}.$$

Then the operators $\mathrm{id} - E_y(y,\mathrm{id})D \in L(Z_0, Z_{0,h})$ *and* $\mathrm{id} - A(y,\bar{y})D \in L(Z_0, Z_{0,h})$ *are uniformly bounded for all* $y \in Y^+$ *with* $\|y\|_{Y^+} \le C_y$*, where* C_y *denotes the constant from Lemma 4.6.*

Proof. Choose an arbitrary $C_y > \|\bar{y}\|_{Y^+}$. Since $E_y(\cdot, \mathrm{id})$ depends only linearly on v we have $A(y,\bar{y}) = E_y((y+\bar{y})/2, \mathrm{id})$ for all $y \in Y^+$. It is therefore enough to consider $E_y(y,\mathrm{id})$ for any $y \in Y^+$ with $\|y\|_{Y^+} \le C_y$.

We have $Z = Z_1 \times Z_2 \times \mathbf{L}^2$, where $Z_1 = \big(L^2(I;V) + P(I;\mathbf{H}_0^1,\mathbf{L}^2)\big)^*$, $Z_2 = \big(L^2(I;L_0^2) \cap H^1(I;(H^{\frac{3}{2}-\varepsilon})^*)\big)$, see Table 4.1.

We show first that $E_y(y,\mathrm{id})D \in L(Z,Z)$ is uniformly bounded and start to consider the divergence component of $z \mapsto E_y(y,\mathrm{id})Dz.$, which is given by $z \mapsto \mathrm{div}(Bz_{\mathrm{div}}) = z_{\mathrm{div}}$ and is thus clearly in $L(Z, Z_2)$. For the initial condition component $z \in Z \mapsto (Bz_{\mathrm{div}})(\cdot,0)$ we obtain the estimate

$$\|(Bz_{\mathrm{div}})(\cdot,0)\|_{\mathbf{L}^2} \le \|Bz_{\mathrm{div}}\|_{C(\bar{I};\mathbf{L}^2)} \le C\|Dz\|_{W(I;\mathbf{H}_0^1)\times\{0\}} \le C\|z\|_Z,$$

since $D \in L(Z,Y)$ by Lemma 4.11 and thus this component is in $L(Z,\mathbf{L}^2)$.

It remains to consider the momentum component of $z \mapsto E_y(y,\mathrm{id})Dz$. Since $L^2(I;\mathbf{H}^{-1}) \hookrightarrow (L^2(I;V) + P(I;\mathbf{H}_0^1,\mathbf{L}^2))^* = Z_1$, it is sufficient to estimate each term in the $L^2(I;\mathbf{H}^{-1})$-norm. Using $\mathbf{H}_0^{-\frac{1}{2}+\varepsilon} \hookrightarrow \mathbf{H}^{-1}$ we obtain

$$\begin{aligned}
&\|(Bz_{\mathrm{div}})_t - \nu\Delta(Bz_{\mathrm{div}}) + b_I(v, Bz_{\mathrm{div}}, \cdot) + b_I(Bz_{\mathrm{div}}, v, \cdot)\|_{L^2(I;\mathbf{H}^{-1})} \\
&\quad \le C\cdot\Big(\|(Bz_{\mathrm{div}})_t\|_{L^2(I;\mathbf{H}_0^{-\frac{1}{2}+\varepsilon})} + \|Bz_{\mathrm{div}}\|_{L^2(I;\mathbf{H}_0^1)} + \|v\|_{L^\infty(I;\mathbf{H}_0^1)}\|Bz_{\mathrm{div}}\|_{L^2(I;\mathbf{H}_0^1)}\Big) \\
&\quad \le C\cdot\Big(\|(z_{\mathrm{div}})_t\|_{L^2(I;H_0^{-\frac{3}{2}+\varepsilon})} + \big(1 + \|v\|_{L^\infty(I;\mathbf{H}_0^1)}\big)\|z_{\mathrm{div}}\|_{L^2(I;L_0^2)}\Big) \le C\|z\|_Z
\end{aligned}$$

with a constant $C > 0$ depending only on $\|y\|_{Y^+}$. Hence, this component is uniformly bounded in $L(Z, L^2(I; \mathbf{H}^{-1})) \hookrightarrow L(Z, Z_1)$ and thus the uniform boundedness of $E_y(y, \mathrm{id})D \in L(Z, Z)$ is shown.

It remains to show that $\mathrm{id} - E_y(y, \mathrm{id})D \in L(Z_0, Z_{0.h})$. Let $z = (z_1, z_{\mathrm{div}}, z_0) \in Z_0$ be arbitrary and set $\tilde{z} = (\tilde{z}_1, \tilde{z}_{\mathrm{div}}, \tilde{z}_0) := (\mathrm{id} - E_y(y, \mathrm{id})D)z$. Then

$$\tilde{z}_{\mathrm{div}} = z_{\mathrm{div}} - \operatorname{div} B z_{\mathrm{div}} = z_{\mathrm{div}} - z_{\mathrm{div}} = 0$$
$$\tilde{z}_0 = z_0 - (B z_{\mathrm{div}})(0), \quad \operatorname{div} \tilde{z}_0 = \operatorname{div} z_0 - z_{\mathrm{div}}(0) = 0,$$

where the last equality follows from $z \in Z_0$. This shows that $\tilde{z} \in Z_{0,h}$. □

Proof. (of Lemma 4.12) As observed in the previous proof we have $A(y, \bar{y}) = E_y((y + \bar{y})/2, \mathrm{id})$. Hence, $A(y, \bar{y}) \in L(Y^+, Z)$ is obvious and it is enough to consider $E_y(y, \mathrm{id})$ for any $y \in Y^+$ with $\|y\|_{Y^+} \leq C_y$.

We proceed as sketched above. Consider

$$E_y(y, \mathrm{id})(h^v, h^p) = (z_1, z_{\mathrm{div}}, z_0)$$

for arbitrary $z = (z_1, z_{\mathrm{div}}, z_0) \in Z_0$. With $D \in L(Z, Y)$ in Lemma 4.11 we make the ansatz

$$(h^v, h^p) = Dz + (h_0^v, h^p),$$

which leads to

$$E_y(y, \mathrm{id})(h_0^v, h^p) = (\mathrm{id} - E_y(y, \mathrm{id})D)z =: \tilde{z} = (\tilde{z}_1, 0, \tilde{z}_0) \in Z_{0,h},$$

since $\mathrm{id} - E_y(y, \mathrm{id})D \in L(Z_0, Z_{0,h})$ by Lemma 4.13.

Hence, we have to show that $E_y(y, \mathrm{id})^{-1} \in L(Z_{0,h}, Y)$. This can be obtained by standard techniques, since $\tilde{z}_0 \in H$ and $\tilde{z}_1 \in (L^2(I; V) + P(I; \mathbf{H}_0^1, \mathbf{L}^2))^*$ by the definition of $Z_{0,h}$. In fact, by using solenoidal test functions and the incompressibility condition, the existence of a unique $h_0^v \in W(I; V)$ with $h_0^v(0) = \tilde{z}_0$ is well known. More precisely, the proof of [22, Proposition 2.1] also holds true for dimension $d = 3$, since $\|y\|_{Y^+} \leq C_y$ implies that the velocity component is uniformly bounded in $L^\infty(I; \mathbf{L}^3) \cap L^4(I; \mathbf{H}_0^1)$. Hence, the proof of [22, Proposition 2.1] yields the invertibility of $E_y(y, \mathrm{id})$ with the estimate $\|h_0^v\|_{W(I;V)} \leq C\|\tilde{z}\|_{Z_{0,h}} \leq C\|z\|_{Z_0}$ for the velocity component.

The uniqueness and $P(I; L_0^2, \mathrm{cl}_{H_0^{-1}}(L_0^2))^*$-regularity of the pressure component h^p can be obtained similarly as in [43, p. 307-308] by utilizing that the linearized equation holds in the sense of distributions. In fact, for $\varphi \in C_0^\infty(\Omega \times [0, T))$ with $\operatorname{div} \varphi(t) = 0$, $t \in [0, T)$, an analogue of (2.10) holds for h^v. Using a test function $\varphi \in C_0^\infty(\Omega \times [0, T])$ in the momentum equation yields by using $h^v \in W(I; V) \hookrightarrow C([0, T]; H)$ after integration by parts in time

$$\begin{aligned}
&\langle h^p, \operatorname{div} \varphi \rangle_{P(I; L_0^2, \mathrm{cl}_{H_0^{-1}}(L_0^2))^*, P(I; L_0^2, \mathrm{cl}_{H_0^{-1}}(L_0^2))} \\
&\quad = -\langle h_0^v, \varphi_t \rangle_{L^\infty(I; \mathbf{L}^2), L^1(I; \mathbf{L}^2)} + (h_0^v(T), \varphi(T))_{\mathbf{L}^2} - (\tilde{z}_0, \varphi(0))_{\mathbf{L}^2} + \nu(\nabla h_0^v, \nabla \varphi)_{L^2(I; \mathbf{L}^2)} \\
&\qquad + b_I(v, h_0^v, \varphi) + b_I(h_0^v, v, \varphi) - \langle \tilde{z}_1, \varphi \rangle_{P(I; \mathbf{H}_0^1, \mathbf{L}^2)^*, P(I; \mathbf{H}_0^1, \mathbf{L}^2)} \\
&\quad \leq C(\|v\|_{Y^+})(\|h_0^v\|_{W(I;V)} + \|\tilde{z}\|_{Z_0})\|\varphi\|_{P(I; \mathbf{H}_0^1, \mathbf{L}^2)} \leq C\|z\|_{Z_0}\|\varphi\|_{P(I; \mathbf{H}_0^1, \mathbf{L}^2)}.
\end{aligned}$$

For $\varphi \in C_0^\infty(\Omega \times I)$ this follows by the calculus of distributions. The extension to $\varphi \in C_0^\infty(\Omega \times [0, T])$ can be shown by using $h^v \in W(I; V) \hookrightarrow C([0, T]; H)$.

By the above estimate the right hand side of the equality can be extended uniquely to an element in $P(I;\mathbf{H}_0^1,\mathbf{L}^2)^*$ and thus to test functions $\varphi \in P(I;\mathbf{H}_0^1,\mathbf{L}^2)$, since $C_0^\infty(\Omega\times[0,T])$ is dense in $P(I;\mathbf{H}_0^1,\mathbf{L}^2)$. Since $\operatorname{div} : P(I;\mathbf{H}_0^1,\mathbf{L}^2) \to P(I;L_0^2,\mathrm{cl}_{H_0^{-1}}(L_0^2))$ is bounded and surjective (see below), we see that $h^p \in P(I;L_0^2,\mathrm{cl}_{H_0^{-1}}(L_0^2))^*$ is uniquely determined and $\|h^p\|_{P(I;L_0^2,\mathrm{cl}_{H_0^{-1}}(L_0^2))^*} \le C\|z\|_{Z_0}$.

It remains to show that $\operatorname{div} : P(I;\mathbf{H}_0^1,\mathbf{L}^2) \to P(I;L_0^2,\mathrm{cl}_{H_0^{-1}}(L_0^2))$ is bounded and surjective. For the boundedness we note that $C^1([0,T];\mathbf{H}_0^1)$ is dense in $P(I;\mathbf{H}_0^1,\mathbf{L}^2)$ and $\operatorname{div} : (C^1([0,T];\mathbf{H}_0^1),\|\cdot\|_{P(I;\mathbf{H}_0^1,\mathbf{L}^2)}) \mapsto (C^1([0,T];L_0^2),\|\cdot\|_{P(I;L_0^2,\mathrm{cl}_{H_0^{-1}}(L_0^2))})$ is bounded. The surjectivity follows by the choice $\varphi(t) = Bq(t)$ and thus $\operatorname{div}\varphi = q$ with $q \in P(I;L_0^2,\mathrm{cl}_{H_0^{-1}}(L_0^2))$ and the Bogovskiĭ operator $B \in L(H_0^{-1},\mathbf{L}^2)\cap L(L^2,\mathbf{H}_0^1)$ from Lemma 4.10.

For the proof of the continuity of $y \in Y^+ \mapsto E_y(y,\mathrm{id}) \in L(Y^+,Z)$ we observe that in $E_y(y_1,\mathrm{id}) - E_y(y_2,\mathrm{id})$ only the trilinar forms give a contribution, since all linear terms cancel out. Therefore, let $v_1, v_2 \in H^1(I;\mathbf{H}_0^1) \hookrightarrow L^\infty(I;\mathbf{H}_0^1)$ and $v_0 \in W(I;\mathbf{H}_0^1)$ be arbitrarily given. Since $L^2(I;\mathbf{H}^{-1}) \hookrightarrow (L^2(I;V) + P(I;\mathbf{H}_0^1,\mathbf{L}^2))^*$ and

$$\|b_I(v_1-v_2,v_0,\cdot) + b_I(v_0,v_1-v_2,\cdot)\|_{L^2(I;\mathbf{H}^{-1})} \le C\|v_1-v_2\|_{L^\infty(I;\mathbf{H}_0^1)}\|v_0\|_{L^2(I;\mathbf{H}_0^1)},$$

the continuity of $y \in Y^+ \mapsto E_y(y,\mathrm{id}) \in L(Y^+,Z)$ follows. □

To complete the proof of assumption 6 of Lemma 3.2 we show finally

Lemma 4.14. *Let the assumptions of Lemma 4.12 hold. Then we have $E_\tau(\bar y,\mathrm{id}) \in L(U,Z_0)$ and $A(y,\bar y)(y-\bar y) \in Z_0$ for all $y \in Y^+$.*

Proof. The action of $E_\tau(\bar y,\mathrm{id})$ is given in (4.3). The Fréchet differentiability of $\tau \in U \mapsto E(\bar y,\tau) \in Z$ at $\tau = \mathrm{id}$ yields $E_\tau(\bar y,\mathrm{id}) \in L(U,Z)$. Now let $h^\tau \in U$ be arbitrary and consider $z = (z_1, z_{\mathrm{div}}, z_0) := E_\tau(\bar y,\mathrm{id})h^\tau$. Then

$$z_{\mathrm{div}}(0) = \mathrm{tr}\,(g_3'(\mathrm{id})[h^\tau]\nabla\bar v(0)) = \mathrm{tr}\,(g_3'(\mathrm{id})[h^\tau]\nabla\widetilde v_0) = -\,\mathrm{tr}\,(\nabla h^\tau\nabla\widetilde v_0) + \operatorname{div}(h^\tau)\operatorname{div}(\widetilde v_0)$$

and on the other hand $z_0 = -\frac{d}{d\tau}\widetilde v_0(\tau)|_{\tau=\mathrm{id}}[h^\tau]$. Hence,

$$\operatorname{div} z_0 = -\operatorname{div}\left(\tfrac{d}{ds}\widetilde v_0(\mathrm{id}+sh^\tau)|_{s=0}\right) = -\tfrac{d}{ds}\operatorname{div}\left(\widetilde v_0(\mathrm{id}+sh^\tau)\right)|_{s=0}$$

and therefore

$$\begin{aligned}\operatorname{div} z_0 &= -\tfrac{d}{ds}\,\mathrm{tr}\left(\nabla\widetilde v_0(\mathrm{id}+sh^\tau)^T(I+sh^{\tau\prime})\right)|_{s=0}\\ &= -\tfrac{d}{ds}\left(\operatorname{div}(\widetilde v_0)\circ(\mathrm{id}+sh^\tau)\right)|_{s=0} - \mathrm{tr}\,(\nabla h^\tau\nabla\widetilde v_0) = -\,\mathrm{tr}\,(\nabla h^\tau\nabla\widetilde v_0)\,,\end{aligned}$$

since $\operatorname{div}(\widetilde v_0) = 0$ on $(\mathrm{id}+sh^\tau)(\Omega_{\mathrm{ref}})$ for all $h^\tau \in U$ with sufficiently small $s\|h^\tau\|_{\mathbf{W}^{1,\infty}} > 0$. This shows that $z_{\mathrm{div}}(0) = \operatorname{div} z_0$ and thus $E_\tau(\bar y,\mathrm{id})h^\tau \in Z_0$ for all $h^\tau \in U$, i.e., $E_\tau(\bar y,\mathrm{id}) \in L(U,Z_0)$ as asserted.

To show that $(z_1,z_{\mathrm{div}},z_0) := A(y,\bar y)(y-\bar y) = E_y((y+\bar y)/2,\mathrm{id})(y-\bar y) \in Z_0$, we note that by (4.4) and with $y = (v,p)$, $\bar y = (\bar v,\bar p)$ it holds

$$z_{\mathrm{div}}(0) = \operatorname{div}(v(0)-\bar v(0)), \quad z_0 = v(0)-\bar v(0)$$

and thus $z_{\mathrm{div}}(0) = \operatorname{div} z_0$ which shows $A(y,\bar y)(y-\bar y) \in Z_0$ as required. □

4.8. Continuity of the solution operator with respect to domain variations. We now show that under our regularity assumptions on the data the continuity requirements from Theorem 3.1 are satisfied, i.e. the mapping

$$\tau \in \mathcal{U} \subset U \mapsto (v(\tau),p(\tau)) \in Y_i$$

is continuous at $\tau = \mathrm{id}$ for $i = 1, 2$. To this end, according to Lemma 3.2, it sufficies to show property (3.3), since assumptions 1.-3. of Theorem 3.1 and assumption 6. of Lemma 3.2 are already proven.

LEMMA 4.15. *Let $d \in \{2, 3\}$ and let Assumption 2.3 hold. Let $(\bar{v}, \bar{p})$ be the solution of (2.21) for $\tau = \mathrm{id}$ and let $(v(\tau), p(\tau))$ be the solution for an arbitrary τ with sufficiently small $\|\tau - \mathrm{id}\|_{\mathbf{W}^{1,\infty}(\Omega_{\mathrm{ref}})}$. Then we have*

$$\|(v(\tau) - \bar{v}, p(\tau) - \bar{p})\|_{Y_1} \to 0 \Longrightarrow \|(v(\tau) - \bar{v}, p(\tau) - \bar{p})\|_{Y_2} \to 0,$$

as $\|\tau - \mathrm{id}\|_{\mathbf{W}^{1,\infty}(\Omega_{\mathrm{ref}})} \to 0$.

Proof. We set $e^v := v(\tau) - \bar{v}$, $e^p = p(\tau) - \bar{p}$.

Concerning the convergence of e^v in $L^\infty(I; \mathbf{L}^3)$ we recall that for sufficiently small displacements $\|\tau - \mathrm{id}\|_{\mathbf{W}^{1,\infty}(\Omega_{\mathrm{ref}})}$ the velocities $v(\tau)$ are uniformly bounded in $L^\infty(I; \mathbf{H}_0^1) \hookrightarrow L^\infty(I; \mathbf{L}^6)$ by Lemma 4.6. Thus, interpolation yields

$$\|e^v\|_{L^\infty(I;\mathbf{L}^3)} \le C\|e^v\|^{\frac{1}{2}}_{L^\infty(I;\mathbf{L}^6)}\|e^v\|^{\frac{1}{2}}_{L^\infty(I;\mathbf{L}^2)} \le C\|e^v\|^{\frac{1}{2}}_{L^\infty(I;\mathbf{L}^2)},$$

such that $\|(e^v, e^p)\|_{Y_1} \to 0$ implies $\|e^v\|_{L^\infty(I;\mathbf{L}^3)} \to 0$ as $\|\tau - \mathrm{id}\|_{\mathbf{W}^{1,\infty}(\Omega_{\mathrm{ref}})} \to 0$.

In order to finish the proof we have to show $\|e^v_t\|_{L^2(I;\mathbf{H}_0^{1-\varepsilon})}, \|e^p\|_{L^2(I;L^2_0)} \to 0$, as $\|(e^v, e^p)\|_{Y_1} \to 0$ and $\|\tau - \mathrm{id}\|_{\mathbf{W}^{1,\infty}(\Omega_{\mathrm{ref}})} \to 0$. Therefore, let $\widetilde{v} := \frac{v+\bar{v}}{2}$, $\bar{y} := (\bar{v}, \bar{p})$, and $y := (v, p)$. Using the linearization (4.4), a short calculation shows that subtracting the equations (2.21) for (v, p) and $(\bar{v}, \bar{p})$ yields with $D_i(\tau) := g_i(\mathrm{id}) - g_i(\tau)$ for $i = 1, 2, 3$

(4.9)
$$\begin{aligned}
&\langle (w, q, w_0), E_y((\widetilde{v}(t), \bar{p}(t)), \mathrm{id})[(e^v(t), e^p(t))]\rangle_{\mathbf{H}_0^1\times L_0^2\times\mathbf{L}^2,(\mathbf{H}_0^1\times L_0^2\times\mathbf{L}^2)^*}\\
&\quad= ((\widetilde{f}(t) \circ_x \tau)g_1(\tau) - \widetilde{f}(t), w)_{\mathbf{L}^2} + (v_t(t), D_1(\tau)w)_{\mathbf{L}^2}\\
&\qquad+ \nu(\nabla v(t), D_2(\tau)\nabla w)_{\mathbf{L}^2} + \widehat{b}(v(t), v(t), w, D_3(\tau)) - (p(t), \mathrm{tr}(D_3(\tau)\nabla w))_{L_0^2},\\
&\qquad+ (\mathrm{tr}(D_3(\tau)\nabla v(t)), q)_{L_0^2} + (w_0, \widetilde{v}_0 \circ \tau - \widetilde{v}_0)
\end{aligned}$$

for all $w \in \mathbf{H}_0^1$, $q \in L_0^2$, $w_0 \in \mathbf{L}^2$, and for a.a. $t \in I$.

For the estimate of the time derivative we test (4.9) with $q = 0$, $w_0 = 0$ and arbitrary $w \in L^2(I; \mathbf{H}_0^1)$. We show that all other terms in (4.9) tend to zero as functionals w.r.t. $w \in L^2(I; \mathbf{H}_0^1)$. We estimate the right hand side of (4.9) as follows.

$$\begin{aligned}
\left|((\widetilde{f} \circ_x \tau)g_1(\tau) - \widetilde{f}, w)_{L^2(I;\mathbf{L}^2)}\right| &\le \|(\widetilde{f} \circ_x \tau)g_1(\tau) - \widetilde{f}\|_{L^2(I;\mathbf{L}^2)}\|w\|_{L^2(I;\mathbf{L}^2)}\\
\left|(v_t, D_1(\tau)w)_{L^2(I;\mathbf{L}^2)}\right| &\le \|v_t\|_{L^2(I;\mathbf{L}^2)}\|w\|_{L^2(I;\mathbf{L}^2)}\|D_1(\tau)\|_{\mathbf{L}^\infty}.
\end{aligned}$$

Furthermore, by using Lemma 2.6 and Hölder's inequality we get

$$\begin{aligned}
\left|\nu(\nabla v, D_2(\tau)\nabla w)_{L^2(I;\mathbf{L}^2)}\right| &\le \nu\|v\|_{L^2(I;\mathbf{H}_0^1)}\|w\|_{L^2(I;\mathbf{H}_0^1)}\|D_2(\tau)\|_{\mathbf{L}^\infty}\\
|\widehat{b}_I(v, v, w, D_3(\tau))| &\le C\|v\|^2_{L^4(I;\mathbf{H}_0^1)}\|w\|_{L^2(I;\mathbf{H}_0^1)}\|D_3(\tau)\|_{\mathbf{L}^\infty}.
\end{aligned}$$

For the last term we have

$$\left|(p, \mathrm{tr}(D_3(\tau)\nabla w))_{L^2(I;L_0^2)}\right| \le C\|p\|_{L^2(I;L_0^2)}\|D_3(\tau)\|_{\mathbf{L}^\infty}\|w\|_{L^2(I;\mathbf{H}_0^1)}.$$

We now estimate the other terms on the left hand side of (4.9). We obtain

$$\left|\nu(\nabla e, \nabla w)_{L^2(I;\mathbf{L}^2)}\right| \le \nu\|e^v\|_{L^2(I;\mathbf{H}_0^1)}\|w\|_{L^2(I;\mathbf{H}_0^1)}$$

and by using Lemma 2.6 and Hölder's inequality

$$\left|b_I(e^v,\widetilde{v},w)+b_I(\widetilde{v},e^v,w)\right|\le C\cdot\left(\|\widetilde{v}\|_{L^\infty(I;\mathbf{H}_0^1)}\|e^v\|_{L^2(I;\mathbf{H}_0^1)}\|w\|_{L^2(I;\mathbf{H}_0^1)}\right).$$

Hence, for $\|\tau-\mathrm{id}\|_{\mathbf{W}^{1,\infty}(\Omega_{\mathrm{ref}})}$ small enough we obtain from (4.9)

$$\begin{aligned}\left|(e_t^v,w)_{L^2(I;\mathbf{L}^2)}\right|\le{}& C\|w\|_{L^2(I;\mathbf{H}_0^1)}(\|\tau-\mathrm{id}\|_{\mathbf{W}^{1,\infty}}+\|e^v\|_{L^2(I;\mathbf{H}_0^1)})\\ &+\|e^p\|_{L^2(I;L_0^2)}\|\operatorname{tr}(\nabla w)\|_{L^2(I;L_0^2)}\\ &+\|(\widetilde{f}\circ_x\tau)g_1(\tau)-\widetilde{f}\|_{L^2(I;\mathbf{L}^2)}\|w\|_{L^2(I;\mathbf{L}^2)}\end{aligned}\tag{4.10}$$

for all $w\in L^2(I;\mathbf{H}_0^1)$, where

$$C=C(\|v\|_{L^\infty(I;\mathbf{H}_0^1)},\|v\|^2_{L^4(I;\mathbf{H}_0^1)},\|p\|_{L^2(I;L_0^2)},\|v_t\|_{L^2(I;\mathbf{L}^2)},\|\bar{v}\|_{L^\infty(I;\mathbf{H}_0^1)})$$

is a constant, which is uniformly bounded due to the Y^+ regularity of the state variables, cf. Lemma 4.6.

Moreover, we have from the incompressibility condition

$$\operatorname{tr}(\nabla e^v)=\operatorname{tr}(D_3(\tau)\nabla v)$$

and therefore with a constant $C>0$

$$\|\operatorname{tr}(\nabla e_t^v)\|_{L^2(I;L_0^2)}=\|\operatorname{tr}(D_3(\tau)\nabla v_t)\|_{L^2(I;L_0^2)}\le C\|v_t\|_{L^2(I;\mathbf{H}_0^1)}\|\tau-\mathrm{id}\|_{\mathbf{W}^{1,\infty}}.$$

Testing (4.10) with $w=e_t^v$ we thus arrive for sufficiently small $\|\tau-\mathrm{id}\|_{\mathbf{W}^{1,\infty}(\Omega_{\mathrm{ref}})}$ at

$$\begin{aligned}\|e_t^v\|^2_{L^2(I;\mathbf{L}^2)}\le{}& C\|v_t-\bar{v}_t\|_{L^2(I;\mathbf{H}_0^1)}(\|\tau-\mathrm{id}\|_{\mathbf{W}^{1,\infty}}+\|e^v\|_{L^2(I;\mathbf{H}_0^1)})\\ &+C\|p-\bar{p}\|_{L^2(I;L_0^2)}\|v_t\|_{L^2(I;\mathbf{H}_0^1)}\|\tau-\mathrm{id}\|_{\mathbf{W}^{1,\infty}}\\ &+\|(\widetilde{f}\circ_x\tau)g_1(\tau)-\widetilde{f}\|_{L^2(I;\mathbf{L}^2)}\|e_t^v\|_{L^2(I;\mathbf{L}^2)}\end{aligned}$$

and thus

$$\|e_t^v\|^2_{L^2(I;\mathbf{L}^2)}\le C\cdot(\|\tau-\mathrm{id}\|_{\mathbf{W}^{1,\infty}}+\|e^v\|_{L^2(I;\mathbf{H}_0^1)})+\|(\widetilde{f}\circ_x\tau)g_1(\tau)-\widetilde{f}\|^2_{L^2(I;\mathbf{L}^2)},$$

with another constant

$$\begin{aligned}C={}&C(\|v\|_{L^\infty(I;\mathbf{H}_0^1)},\|v\|^2_{L^4(I;\mathbf{H}_0^1)},\|p\|_{L^2(I;L^2)},\|\bar{p}\|_{L^2(I;L_0^2)},\\ &\|v_t\|_{L^2(I;\mathbf{H}_0^1)},\|\bar{v}\|_{L^\infty(I;\mathbf{H}_0^1)},\|\bar{v}_t\|_{L^2(I;\mathbf{H}_0^1)}).\end{aligned}$$

Therefore, using the uniform Y^+ boundedness of the state variables from Lemma 4.6 and the regularity assumptions on $\widetilde{f}$, we have shown

$$\|e_t^v\|_{L^2(I;\mathbf{L}^2)}\to 0,$$

as $\|(e^v,e^p)\|_{Y_1}\to 0$ and $\|\tau-\mathrm{id}\|_{\mathbf{W}^{1,\infty}(\Omega_{\mathrm{ref}})}\to 0$. Since $\|e_t\|_{L^2(I;\mathbf{H}_0^1)}\le C$, interpolation yields

$$\|e_t^v\|_{L^2(I;\mathbf{H}_0^{1-\varepsilon})}\to 0,\tag{4.11}$$

as $\|(e^v,e^p)\|_{Y_1}\to 0$ and $\|\tau-\mathrm{id}\|_{\mathbf{W}^{1,\infty}(\Omega_{\mathrm{ref}})}\to 0$.

For $\|\tau - \mathrm{id}\|_{\mathbf{W}^{1,\infty}(\Omega_{\mathrm{ref}})}$ sufficiently small the pressure term on the left hand side of (4.9) satisfies exactly as in (4.10) the estimate

$$\begin{aligned}\left|(\nabla e^p, w)_{L^2(I;\mathbf{L}^2)}\right| &= \left|(e^p, \mathrm{tr}(\nabla w))_{L^2(I;L_0^2)}\right| \\ &\le C \cdot (\|\tau - \mathrm{id}\|_{\mathbf{W}^{1,\infty}} + \|e^v\|_{L^2(I;\mathbf{H}_0^1)})\|w\|_{L^2(I;\mathbf{H}_0^1)} \\ &\quad + (\|e_t^v\|_{L^2(I;\mathbf{L}^2)} + \|(\widetilde{f} \circ_x \tau)g_1(\tau) - \widetilde{f}\|_{L^2(I;\mathbf{L}^2)})\|w\|_{L^2(I;\mathbf{L}^2)},\end{aligned}$$

which shows $\|\nabla e^p\|_{L^2(I;\mathbf{H}^{-1})} \to 0$ as $\|(e^v, e^p)\|_{Y_1} \to 0$ and $\|\tau - \mathrm{id}\|_{\mathbf{W}^{1,\infty}(\Omega_{\mathrm{ref}})} \to 0$. By [43, Prop. I.1.2] we have $\|e^p(t)\|_{L_0^2} \le C(\Omega_{\mathrm{ref}})\|\nabla e^p(t)\|_{\mathbf{H}^{-1}}$ for all $t \in I$. Hence we conclude that $\|e^p\|_{L^2(I;L_0^2)} \to 0$ as $\|(e^v, e^p)\|_{Y_1} \to 0$ and $\|\tau - \mathrm{id}\|_{\mathbf{W}^{1,\infty}(\Omega_{\mathrm{ref}})} \to 0$. ☐

5. Differentiability of the reduced objective function. We deduce finally from Theorem 4.3 differentiability properties of objective functionals with respect to domain transformations $\tau \in U(\Omega_{\mathrm{ref}})$. To this end, consider as in Section 2.1 a shape optimization problem

$$\min \quad \widetilde{J}(\widetilde{y}, \Omega) \quad \text{s.t.} \quad \widetilde{E}(\widetilde{y}, \Omega) = 0, \quad \Omega \in \mathcal{O}_{\mathrm{ad}}, \tag{2.1}$$

where $\widetilde{y} = (\widetilde{v}, \widetilde{p})$ and $\widetilde{E}(\widetilde{y}, \Omega) = 0$ are the Navier-Stokes equations (2.5). By applying the mapping method, we obtain the equivalent problem on the reference domain Ω_{ref}

$$\min \quad J(y, \tau) \quad \text{s.t.} \quad E(y, \tau) = 0, \quad \tau \in T_{\mathrm{ad}}, \tag{2.4}$$

where $y = (v, p)$ and $E(y, \tau) = 0$ is given by (2.21) and T_{ad} is a set as given in Assumption 2.3 or a closed subset of it.

We state now sufficient conditions for an objective function $J : Y^+(\Omega_{\mathrm{ref}}) \times U(\Omega_{\mathrm{ref}}) \to \mathbb{R}$ such that the reduced objective function

$$\tau \in \mathcal{U} \subset U \mapsto j(\tau) := J((v, p)(\tau), \tau) \in \mathbb{R} \tag{5.1}$$

is Fréchet differentiable in a neighbourhood $\mathcal{U} \subset U$ of id. We note that

$$\tau_1, \tau_2 \in T_{\mathrm{ad}} \text{ with } \tau_1(\Omega_{\mathrm{ref}}) = \tau_2(\Omega_{\mathrm{ref}}) \Longrightarrow j(\tau_1) = j(\tau_2),$$

since by (2.1) the objective function depends only on the domain.

Theorem 4.3, i) shows the differentiability of $\tau \in U \mapsto (v(\tau), p(\tau)) \in Y$ at $\tau = \mathrm{id}$ as well as the differentiability to the slightly weaker space $\left(L^2(I; \mathbf{H}_0^1(\Omega_{\mathrm{ref}})) \cap C(\overline{I}; \mathbf{L}^2(\Omega_{\mathrm{ref}}))\right) \times \left(L^2(I; L_0^2(\Omega_{\mathrm{ref}})) + (W^{1,1}(I; \mathrm{cl}_{H_0^{-1}}(L_0^2(\Omega_{\mathrm{ref}}))))^*\right)$ at $\tau = \widehat{\tau}$ for any $\widehat{\tau} \in T_{\mathrm{ad}}$. We can therefore apply the chain rule.

For practical computations it is most convenient to evaluate the derivative on the physical domain $\widehat{\Omega} := \widehat{\tau}(\Omega_{\mathrm{ref}})$ at id. On the other hand, for optimization methods it is preferable to work on a fixed reference domain Ω_{ref} and to evaluate the derivative at arbitrary $\widehat{\tau} \in T_{\mathrm{ad}}$. Both can easily be accomplished, as the next theorem will show.

To state the theorem, let $\widehat{\tau} \in T_{\mathrm{ad}}$ be arbitrary, let $\widehat{\Omega} = \widehat{\tau}(\Omega_{\mathrm{ref}})$ and denote by $J(\widetilde{y}, \widetilde{\tau}; \widehat{\Omega})$ and $E(\widetilde{y}, \widetilde{\tau}; \widehat{\Omega}) = 0$ the objective function and state equation in (2.4), if $\widehat{\Omega}$ instead of Ω_{ref} is used as reference domain. Finally, let $U(\widehat{\Omega}) = (\mathbf{W}^{1,\infty} \cap \mathbf{W}^{1+s,r})(\widehat{\Omega})$, $\mathcal{U}(\widehat{\Omega}) \subset U(\widehat{\Omega})$ be a sufficiently small neighborhood of id and denote by

$$\widetilde{\tau} \in \mathcal{U}(\widehat{\Omega}) \mapsto y(\widetilde{\tau}; \widehat{\Omega}) = (v(\widetilde{\tau}; \widehat{\Omega}), p(\widetilde{\tau}; \widehat{\Omega})) \in Y^+(\widehat{\Omega}) \tag{5.2}$$

the bounded solution map of $E(\widetilde{y}, \widetilde{\tau}; \widehat{\Omega}) = 0$ according to Lemma 4.6.

THEOREM 5.1. *Let with the previous notations* $\widehat{\tau} \in T_{\mathrm{ad}}$ *be arbitrary,* $\widehat{\Omega} = \widehat{\tau}(\Omega_{\mathrm{ref}})$ *and let* $\bar{y} \in Y^+(\widehat{\Omega})$ *be the physical state on* $\widehat{\Omega}$*, i.e.,* $E(\bar{y}, \mathrm{id}; \widehat{\Omega}) = 0$*. Finally, let*

$$(\widetilde{y}, \widetilde{\tau}) \in (\mathcal{Y}, \|\cdot\|_{Y(\widehat{\Omega})}) \times U(\widehat{\Omega}) \mapsto J(\widetilde{y}, \widetilde{\tau}; \widehat{\Omega})$$

be Fréchet-differentiable at $(\bar{y}, \mathrm{id})$ *on bounded sets* $\bar{y} \ni \mathcal{Y} \subset Y^+(\widehat{\Omega})$ *with* $J_y(\bar{y}, \mathrm{id}; \widehat{\Omega}) \in Y^*$*. Then the following holds.*

i) The reduced objective function

$$\widetilde{\tau} \in \mathcal{U}(\widehat{\Omega}) \subset U(\widehat{\Omega}) \mapsto \widehat{j}(\widetilde{\tau}) := J(y(\widetilde{\tau}; \widehat{\Omega}), \widetilde{\tau}; \widehat{\Omega})$$

corresponding to the reference domain $\widehat{\Omega}$ *is Fréchet-differentiable at* $\widetilde{\tau} = \mathrm{id}$ *and the derivative is given by the chain rule, i.e.,*

$$\widehat{j}(\mathrm{id})(\widetilde{\tau} - \mathrm{id}) = J'(\bar{y}, \mathrm{id})[y'(\mathrm{id}; \widehat{\Omega})(\widetilde{\tau} - \mathrm{id}), \widetilde{\tau} - \mathrm{id}], \tag{5.3}$$

where $y'(\mathrm{id}; \widehat{\Omega}) = (v'(\mathrm{id}; \widehat{\Omega}), p'(\mathrm{id}; \widehat{\Omega}))$ *can be computed as in Theorem 4.3, i).*

ii) With a sufficiently small neighborhood $\mathcal{U}(\Omega_{\mathrm{ref}}) \subset U(\Omega_{\mathrm{ref}})$ *of* $\widehat{\tau}$ *the reduced objective function*

$$\tau \in \mathcal{U}(\Omega_{\mathrm{ref}}) \subset U(\Omega_{\mathrm{ref}}) \mapsto j(\tau) := J(y(\tau), \tau)$$

corresponding to the reference domain Ω_{ref} *is Fréchet-differentiable at* $\tau = \widehat{\tau}$ *and*

$$j'(\bar{\tau})[h^\tau] = \widehat{j}'(\mathrm{id})[h^\tau \circ \bar{\tau}^{-1}] \quad \forall\, h^\tau \in U(\Omega_{\mathrm{ref}}).$$

Proof. i): As observed above, the solution map (5.2) is bounded and thus we find a bounded neighborhood $\mathcal{Y} \subset Y^+(\widehat{\Omega})$ of $\bar{y}$ with $y(\widetilde{\tau}; \widehat{\Omega}) \in \mathcal{Y}$ for all $\widetilde{\tau} \in \mathcal{U}(\widehat{\Omega})$.

Moreover, by Theorem 4.3, i) (see the proof of Theorem 4.3, ii)), the solution map (5.2) is Fréchet-differentiable at $\widetilde{\tau} = \mathrm{id}$ as a map $\widetilde{\tau} \in \mathcal{U}(\widehat{\Omega}) \mapsto y(\widetilde{\tau}; \widehat{\Omega}) \in Y(\widehat{\Omega})$. This yields as in the proof of the chain rule for $\|\widetilde{\tau} - \mathrm{id}\|_{U(\widehat{\Omega})} \to 0$

$$\begin{aligned}
&\widehat{j}(\widetilde{\tau}) - \widehat{j}(\mathrm{id}) - J'(\bar{y}, \mathrm{id})[y'(\mathrm{id}; \widehat{\Omega})(\widetilde{\tau} - \mathrm{id}), \widetilde{\tau} - \mathrm{id}] \\
&\quad = J(y(\widetilde{\tau}; \widehat{\Omega}), \widetilde{\tau}; \widehat{\Omega}) - J(\bar{y}, \mathrm{id}; \widehat{\Omega}) - J'(\bar{y}, \mathrm{id})[y'(\mathrm{id}; \widehat{\Omega})(\widetilde{\tau} - \mathrm{id}), \widetilde{\tau} - \mathrm{id}] \\
&\quad = o(\|y(\widetilde{\tau}; \widehat{\Omega}) - \bar{y}\|_{Y(\widehat{\Omega})} + \|\widetilde{\tau} - \mathrm{id}\|_{U(\widehat{\Omega})}) \\
&\qquad + J_y(\bar{y}, \mathrm{id})[y(\widetilde{\tau}; \widehat{\Omega}) - \bar{y} - y'(\mathrm{id}; \widehat{\Omega})(\widetilde{\tau} - \mathrm{id})] \\
&\quad = o(\|\widetilde{\tau} - \mathrm{id}\|_{U(\widehat{\Omega})}).
\end{aligned}$$

ii): Since $\tau \circ \widehat{\tau}^{-1} \in \mathcal{U}(\widehat{\Omega})$ for all $\tau \in \mathcal{U}(\Omega_{\mathrm{ref}})$ with $\|\tau - \widehat{\tau}\|_{U(\Omega_{\mathrm{ref}})} \leq \rho$, $\rho > 0$ small enough, and $\tau(\Omega_{\mathrm{ref}}) = \tau \circ \widehat{\tau}^{-1}(\widehat{\Omega})$, we have

$$\begin{aligned}
j(\tau) - j(\widehat{\tau}) = \widehat{j}(\tau \circ \widehat{\tau}^{-1}) - \widehat{j}(\mathrm{id}) &= \widehat{j}'(\mathrm{id})[(\tau - \widehat{\tau}) \circ \widehat{\tau}^{-1}] + o(\|(\tau - \widehat{\tau}) \circ \widehat{\tau}^{-1}\|_{U(\widehat{\Omega})}) \\
&= \widehat{j}'(\mathrm{id})[(\tau - \widehat{\tau}) \circ \widehat{\tau}^{-1}] + o(\|\tau - \widehat{\tau}\|_{U(\Omega_{\mathrm{ref}})}),
\end{aligned}$$

where we have used Lemma 4.5 in the last step. ◻

REMARK 5.2. *A direct application of the chain rule together with Theorem 4.3 would require stronger differentiability properties for* $J(\cdot; \widehat{\Omega})$*. The additional boundedness of the state in* $Y^+(\widehat{\Omega})$ *makes it possible to require the differentiability only on bounded sets in* $Y^+(\widehat{\Omega})$*.*

REMARK 5.3. *Instead of the sensitivity based formula* (5.3) *standard arguments yield an adjoint representation of* $\widehat{j}'(\mathrm{id})$*, see [7, sec. 3.4 and 3.5] and [30, sec. 3.3].*

Example: Mean drag of a body and vortex reduction. We consider the mean drag around a simply connected body B. Let $\Gamma_B \subset \partial\Omega$ be the boundary B and $\phi \in \mathbb{R}^d$ be a unit vector in the mean flow direction. Then the mean value of the drag on Γ_B is given by

$$c_d := \frac{1}{T}\int_0^T \int_{\Gamma_B} n \cdot \sigma(\widetilde{v}, \widetilde{p}) \cdot \phi \, dS \, dt$$

with the unit outer normal n and the stress tensor $\sigma(\widetilde{v}, \widetilde{p}) = \nu(\nabla\widetilde{v} + \nabla\widetilde{v}^T) - \widetilde{p}I$. Let $\Phi \in C^1(\mathbb{R}^d)^d$ be a function with

$$\Phi|_{\Gamma_B} = \phi, \quad \Phi|_{\partial\Omega\setminus\Gamma_B} = 0 \quad \forall\, \Omega \in \mathcal{O}_{ad}. \tag{5.4}$$

Then the mean drag can alternatively be computed by the formula

$$\begin{aligned} J((\widetilde{v}, \widetilde{p}); \Omega) := {} & \frac{1}{T}\int_0^T \int_\Omega \Big(((\widetilde{v}\cdot\nabla)\widetilde{v} - \widetilde{f})^T \Phi - \widetilde{p}\,\mathrm{div}\,\Phi + \nu\nabla\widetilde{v} : \nabla\Phi \Big) \, d\widetilde{x}\, dt \\ & + \frac{1}{T}\int_\Omega (\widetilde{v}(\widetilde{x}, T) - \widetilde{v}_0(\widetilde{x}))\Phi \, d\widetilde{x}, \end{aligned}$$

see e.g. [23], where we have integrated the time derivative. Transformation to the reference domain yields finally

$$\begin{aligned} J((v,p), \tau) = \frac{1}{T}\int_0^T & \Big(\widehat{b}(v(t), v(t), \Phi\circ\tau, g_3(\tau)) + \nu(\nabla v, g_2(\tau)\nabla(\Phi\circ\tau))_{\mathbf{L}^2(\Omega_{\mathrm{ref}})} \\ & - (p(t), \mathrm{tr}(g_3(\tau)\nabla(\Phi\circ\tau)))_{L_0^2(\Omega_{\mathrm{ref}})} - \int_{\Omega_{\mathrm{ref}}} (\widetilde{f}(t) \circ_{\widetilde{x}} \tau)^T (\Phi\circ\tau) g_1(\tau) dx \Big) \, dt \\ & + \frac{1}{T}(v(T) - \widetilde{v}_0\circ\tau, (\Phi\circ\tau)g_1(\tau))_{\mathbf{L}^2(\Omega_{\mathrm{ref}})}. \end{aligned}$$

Let $\widehat{\tau} \in T_{\mathrm{ad}}$ be arbitrary. To show the Fréchet-differentiability of the reduced objective function $j(\tau)$ at $\tau = \widehat{\tau}$, we could apply Theorem 5.1, but we can also immediately use Theorem 4.3, ii) and the chain rule. In fact, since $\Phi \in C^1(\mathbb{R}^d)^d$, (5.4) yields $\mathrm{div}\,\Phi \in L_0^2(\tau(\Omega_{\mathrm{ref}}))$ and thus $\mathrm{tr}(g_3(\tau)\nabla(\Phi\circ\tau)) \in L_0^2(\Omega_{\mathrm{ref}})$, we also have $\mathrm{tr}(g_3(\tau)\nabla(\Phi\circ\tau)) \in L^2(I; L_0^2(\Omega_{\mathrm{ref}})) \cap W^{1,1}(I; L_0^2(\Omega_{\mathrm{ref}}))$. Now, it is easy to verify that

$$\begin{aligned} J : (v, p, \tau) \in {} & (L^2(I; \mathbf{H}_0^1(\Omega_{\mathrm{ref}})) \cap C(\overline{I}; \mathbf{L}^2(\Omega_{\mathrm{ref}}))) \\ & \times \big(L^2(I; L_0^2(\Omega_{\mathrm{ref}})) + (W^{1,1}(I; L_0^2(\Omega_{\mathrm{ref}})))^* \big) \times U \mapsto \mathbb{R} \end{aligned}$$

is Fréchet-differentiable. Hence, the chain rule and Theorem 4.3 yield the Fréchet-differentiability of $\tau \in U \mapsto j(\tau) \in \mathbb{R}$ at $\tau = \widehat{\tau}$.

An adjoint representation of the reduced derivative $j'(\tau)$ can now be derived by standard techniques and integration by parts leads for sufficiently regular problems to the Hadamard-Zolésio boundary representation, see for example [7, 8].

Of course, other objective functions than drag can be considered as well, the only requirement being that they define a differentiable mapping from Y to $\mathbb{R}$. In the context of vortex reduction one can, e.g., consider [21, 26]

$$J_1(\widetilde{v}, \widetilde{p}; \Omega) := \int_0^T\!\!\int_{\Omega_0} h\big(\det(\nabla\widetilde{v}(t,x))\big) dx dt, \quad J_2(\widetilde{v}, \widetilde{p}; \Omega) := \int_0^T\!\!\int_{\Omega_0} \big|\mathrm{curl}(\widetilde{v}(t,x))\big|^2 dx dt,$$

where $\Omega_0 \subset \Omega$ denotes a Lipschitz subset over which vortex reduction is desired and $h\colon \mathbb{R} \to \mathbb{R}^+$ defines a suitable differentiable function. In analogy to the mean drag objective functional one derives corresponding formulations of $J_1((v,p),\tau)$ and $J_2((v,p),\tau)$ on the reference domain by applying the transformation rule for integrals. While the space Y is sufficiently strong to obtain continuous differentiability of $J_2\colon Y \times U \to \mathbb{R}$ for dimension $d = 2, 3$, the nonlinearity of J_1 caused by the determinant and the smoothing function h requires a refined investigation. In [26, Ch. 2] this function is given by $h(s) = 0$ for $s \leq 0$ and $h(s) = \frac{s^3}{s^2+1}$ for $s > 0$. An application of differentiability theory for Nemytskii operators, e.g. [17], yields the differentiability of $\widetilde{v} \in L^2(I; H^1(\Omega)) \mapsto h\big(\det(\nabla\widetilde{v})\big) \in L^1(I; L^1(\Omega))$ for dimension $d = 2$ and therefore also the differentiability of $J_1\colon Y \times U \to \mathbb{R}$ for $d = 2$.

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