Optimal Control of Nonlinear Hyperbolic Conservation Laws by On/Off-Switching

Sebastian Pfaff[a] and Stefan Ulbrich[a*]

[a] *Department of Mathematics, TU Darmstadt, Germany*

This paper studies the differentiability properties of the control-to-state mapping for entropy solutions to a scalar hyperbolic conservation law on $\mathbb{R}$ with respect to the switching times of an on/off-control. The switching times between on-modes and off-modes are the control variables of the considered optimization problem, where a general tracking-type functional is minimized.

We investigate the differentiability of the reduced objective function, also in the presence of shocks. We show that the state $y(\bar{t},\cdot)$ at some observation time $\bar{t}$ depends differentiably on the switching times in a generalized sense that implies total differentiability for the composition with a tracking functional. Furthermore, we present an adjoint-based formula for the gradient of the reduced objective functional with respect to the switching times.

Keywords: optimal control; scalar conservation law; network

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1. Introduction

This paper is concerned with the optimal control of entropy solutions of a scalar conservation law

$$\begin{aligned} y_t + f(y)_x &= g(\cdot, y), \quad (t,x) \in]0,\infty[\times \mathbb{R}, \\ y(0,\cdot) &= u_0, \quad x \in \mathbb{R}, \end{aligned} \tag{1.1}$$

which contains an on/off-switching control at $x = 0$, i.e., in the off-state the flux across $x = 0$ is zero leading to a decoupling of (1.1) into two conservation laws and in the on-state the conservation law (1.1) holds on the whole real line $\mathbb{R}$. Such on/off-controls are essential for modeling flows on networks, for example traffic flow involving traffic lights and gas-/water-networks involving valves. They can be seen as node conditions on a simple network consisting of a single node with one incoming and one outgoing edge. However, the results of this paper are also relevant for larger networks with on/off-controls.

We develop a sensitivity and adjoint calculus for objective functionals of the form

$$J(y(\sigma)) := \int_a^b \psi(y(\bar{t},x;\sigma), y_d(x))\, \mathrm{d}x \tag{1.2}$$

*Corresponding author. Email: ulbrich@mathematik.tu-darmstadt.de

with respect to the switching times $\sigma = (\sigma_{\text{on}}^0, \sigma_{\text{off}}^1, \sigma_{\text{on}}^1, \cdots, \sigma_{\text{on}}^{n_\sigma}, \sigma_{\text{off}}^{n_\sigma+1})$. Here, $\psi \in \mathrm{C}^{1,1}_{\text{loc}}(\mathbb{R}^2)$ and $y_d \in \mathrm{BV}(a,b)$ is a desired state.

In [48] a sensitivity calculus for the Cauchy problem with respect to the initial data u_0 was presented that deals with shocks in the entropy solution and also allows for explicit shifts of shock generating discontinuities in the initial data. This approach was extended in [42] to initial-boundary value problems. In this paper we build on the aforementionend works and develop a sensitivity calculus for the on/off-problem. Furthermore, we present an adjoint based formula for the reduced gradient $\frac{\mathrm{d}}{\mathrm{d}\sigma} J(y(\sigma))$ in the flavor of [49]. Our results are valid for arbitrary shock formations in the solution.

It is well known that weak solutions to hyperbolic conservation laws are in general not unique. The physically meaningful solution among them is called *entropy solution* and can be characterized by an entropy condition, see e.g. [33].

Even for smooth initial data, entropy solutions may develop discontinuities, so called shocks, cf. [9], that lead to the issue that the control-to-state mapping $u_0 \mapsto y(\bar{t}, \cdot; u_0)$ for the Cauchy problem is only differentiable with respect to the weak topology of measures. This topology is not strong enough to directly imply the Fréchet-differentiability of the reduced objective (1.2).

Despite these difficulties, the optimal control of conservation laws has been studied intensively in recent years. The existence of optimal controls for the Cauchy and the initial-boundary value problem is well studied in the literature, e.g. [2, 3, 46, 47].

Several generalized notions of differentiability for the control-to-state mapping have been considered, see [7, 10, 11, 13, 16, 47, 48]. In the present work we follow the ideas of [47, 48] and use the concept of *shift-differentiability* introduced therein. A useful tool for establishing shift-differentiability is the theory of generalized characteristics by Dafermos [19]. The shock sensitivity is computed via an appropriate adjoint state which is based on reversible solutions to linear transport equations with discontinuous coefficients [8]. This adjoint calculus can also be used to derive an easily computable formula for the gradient of the reduced objective function, see also [22, 23, 49].

Typical applications for networks of conservation laws are traffic flow modeling and gas pipelines. Traffic networks in the context of the LWR-model [36, 45] and also the Aw-Rascle-model [4] have been discussed by several authors, see for example [12, 14, 21, 26, 30, 31]. In [5, 17, 25, 32] networks of pipelines are considered. Moreover there are several further applications, such as supply chain management [24] or population modeling [15]. In many of these network models on/off-switching devices and their control, which are considered in this paper, are relevant (e.g., valves or traffic lights).

The notion of a "switched control" in the context of hyperbolic conservation laws was discussed in [28], where also switching in flux function and the source term was considered. Further considerations on switching controls can be found in [1, 27, 29].

Our results provide the possibility to apply gradient-based optimization methods to the infinite dimensional optimal control problem for switching times of on-/off-controls and to derive optimality conditions for it. This in turn can form the basis for the development of suitable numerical methods for such type of problems. Furthermore, the extension of our results to systems of conservation laws is of great interest, especially in the context of the optimal control of valves in gas networks or water networks or of traffic lights for more involved traffic models by systems of conservation laws.

The paper is organized as follows. In §2 we introduce the considered general on/off-switching problem and illustrate it by the example of a traffic light in the context of the LWR-model. In §3 we collect results on the well-posedness for this problem and structural properties of the corresponding solution. The main results of our paper will be presented in §4, where we state the shift-differentiability of the control-to-state mapping and the adjoint-based formula for the Fréchet-derivative of the reduced objective function. Those results were already announced in [43]. The proofs of the theorems are postponed to §5.

2. Formulation of the on/off-switching problem

We consider the Cauchy problem for convex scalar conservation laws with source terms on $\mathbb{R}$. We choose the initial data and the source term in such a way that the solution stays inside a certain range $[y_{\min}, y_{\max}]$, which we set to $[0,1]$ for simplicity. Furthermore, we assume that the flux function vanishes at the end points of this interval.

We augment the model by the possibility to suspend the flux across the point $x=0$ for a certain time. The on/off-switching problem splits the considered time interval $]0,T[$ into *on-phases* $]\sigma_{\text{on}}^{i-1}, \sigma_{\text{off}}^{i}[$, $i=1,\dots,n_\sigma+1$ and *off-phases* $]\sigma_{\text{off}}^{i}, \sigma_{\text{on}}^{i}[$, $i=1,\dots,n_\sigma$, where the incoming flux at $x=0$ is or is not allowed to cross, respectively. This is done by imposing an artificial boundary at $x=0$ during the off-phases.

Before we formally define the solution to a on/off-switching problem, we briefly recall the notion of a solution to initial (-boundary) value problems.

2.1 *Solutions to initial (-boundary) value problems*

An initial-boundary value problem (IBVP) on $\Omega =]\mathfrak{a},\mathfrak{b}[$ is given by

$$\begin{aligned}
y_t + f(y)_x &= g(\cdot,y), && \text{on } \Omega_T, && (2.1a)\\
y(0,\cdot) &= u_0, && \text{on } \Omega, && (2.1b)\\
y(\cdot,a+) &= u_{B,\mathfrak{a}}, && \text{in the sense of (2.3a)} \quad (\text{if } \mathfrak{a} > -\infty), && (2.1c)\\
y(\cdot,b-) &= u_{B,\mathfrak{b}}, && \text{in the sense of (2.3b)} \quad (\text{if } \mathfrak{b} < \infty), && (2.1d)
\end{aligned}$$

where $\Omega_T :=]0,T[\times \Omega$. Usually, we are interested in entropy solutions of (2.1), namely solutions that satisfy (2.1a)-(2.1b) in the sense of [33], i.e.

$$\begin{aligned}
(\eta_c(y))_t + (q_c(y))_x - \eta_c'(y)g(\cdot,y) &\le 0, && \text{in } \mathcal{D}'(\Omega_T),\\
\operatorname*{esslim}_{t\to 0+} \|y(t,\cdot) - u_0\|_{1,\Omega\cap]-R,R[} &= 0, && \text{for all } R > 0
\end{aligned}$$

holds for every (Kružkov-) entropy $\eta_c(\lambda) := |\lambda - c|$, $c \in \mathbb{R}$, and associated entropy flux $q_c(\lambda) := \operatorname{sgn}(\lambda - c)(f(\lambda) - f(c))$. In order to get a well posed problem, the boundary conditions (2.1c), (2.1d) have to be understood in the sense of [6], that is

$$\begin{aligned}
\min_{k\in I(y(\cdot,\mathfrak{a}+),u_{B,\mathfrak{a}})} \operatorname{sgn}(u_{B,\mathfrak{a}} - y(\cdot,\mathfrak{a}+))(f(y(\cdot,\mathfrak{a}+)) - f(k)) &= 0, \quad \text{a.e. on }]0,T[, && (2.3a)\\
\min_{k\in I(y(\cdot,\mathfrak{b}-),u_{B,\mathfrak{b}})} \operatorname{sgn}(y(\cdot,\mathfrak{b}-) - u_{B,\mathfrak{b}})(f(y(\cdot,\mathfrak{b}-)) - f(k)) &= 0, \quad \text{a.e. on }]0,T[, && (2.3b)
\end{aligned}$$

with $I(\alpha,\beta) := [\min(\alpha,\beta), \max(\alpha,\beta)]$, see also [20, 35, 37, 39].

2.2 *Solutions to on/off-switching problems*

We will work in the setting $\sigma = (\sigma_{\text{on}}^0, \sigma_{\text{off}}^1, \sigma_{\text{on}}^1, \dots, \sigma_{\text{on}}^{n_\sigma}, \sigma_{\text{off}}^{n_\sigma+1}) \in \Sigma$, where

$$\Sigma := \left\{ \nu \in \mathbb{R}^{2(n_\sigma+1)} \;:\; 0 = \nu_1 < \nu_2 < \cdots < \nu_{2n_\sigma+1} < \nu_{2n_\sigma+2} = T \right\}. \tag{2.4}$$

Of course, the presented analysis is also applicable if one considers the case where the first and/or the final phase is an off-phase.

A solution y of an on/off-switching problem (OOSP) on $\Omega_T :=]0, T[\times\mathbb{R}$ is determined as follows. During the i-th on-phase, y solves a Cauchy problem on $\Omega_{\text{on},i} :=]\sigma_{\text{on}}^{i-1}, \sigma_{\text{off}}^i[\times\mathbb{R}$ with initial data

$$u_0 = y(\sigma_{\text{on}}^{i-1}-, \cdot), \quad i = 2, \dots, n_\sigma + 1$$

in the sense of (2.2), where $y(\sigma_{\text{on}}^{i-1}-, \cdot)$ is the final state of the previous off-phase.

For the i-th off-phase we consider the restrictions y_1 and y_2 of y to the incoming and outgoing arc $I_1 := \mathbb{R}^-$and $I_2 := \mathbb{R}^+$. The restriction y_1 is the solution of an IBVP on $\Omega_{\text{off},i}^1 :=]\sigma_{\text{off}}^i, \sigma_{\text{on}}^i[\times I_1$ with initial value $y(\sigma_{\text{off}}^i-, \cdot)$ and boundary data $u_{B,0} \equiv 0$. Analogously, y_2 solves an IBVP on $\Omega_{\text{off},i}^2 :=]\sigma_{\text{off}}^i, \sigma_{\text{on}}^i[\times I_2$ with $u_{B,0} \equiv 1$. For the first on-phase, i.e. the first IVP, the initial data are given by some function u_I. The on/off-switching problem can then be formulated in the following way.

$$\begin{aligned}
y_t + f(y)_x &= g(\cdot, y) && \text{on } \Omega_{\text{on},i+1}, && i = 0, \dots, n_\sigma, && \text{(2.5a)}\\
(y_j)_t + f(y_j)_x &= g(\cdot, y_j) && \text{on } \Omega_{\text{off},i}^1 \text{ and } \Omega_{\text{off},i}^2, && i = 1, \dots, n_\sigma,, && \text{(2.5b)}\\
y(0, \cdot) &= u_I && \text{on } I, && && \text{(2.5c)}\\
y(\sigma_{\text{on}}^i, \cdot)\big|_{I_j} &= y_j(\sigma_{\text{on}}^i-, \cdot) && \text{on } \mathbb{R}^- \text{ and } \mathbb{R}^+, && i = 1, \dots, n_\sigma,, && \text{(2.5d)}\\
y_j(\sigma_{\text{off}}^i, \cdot) &= y(\sigma_{\text{off}}^i-, \cdot)\big|_{I_j} && \text{on } \mathbb{R}^- \text{ and } \mathbb{R}^+, && i = 1, \dots, n_\sigma,, && \text{(2.5e)}\\
y_1(\cdot, 0-) &= 0 \text{ (in the sense of (2.3a))} && \text{on }]\sigma_{\text{off}}^i, \sigma_{\text{on}}^i[, && i = 1, \dots, n_\sigma, && \text{(2.5f)}\\
y_2(\cdot, 0+) &= 1 \text{ (in the sense of (2.3b))} && \text{on }]\sigma_{\text{off}}^i, \sigma_{\text{on}}^i[, && i = 1, \dots, n_\sigma. && \text{(2.5g)}
\end{aligned}$$

As we will see, the boundary conditions (2.5f)–(2.5g) ensure the flux across $x = 0$ to be equal to zero during off-phases. Condition (2.5d) and (2.5e) ensure, that $t \mapsto y(t, \cdot) \in \mathrm{L}_{\text{loc}}^1(\mathbb{R})$ is continuous, even between consecutive phases.

2.3 *Example: A traffic light on a single road*

A typical example for a single conservation law with on/off-switching control is a traffic light on a unidirectional road. In the LWR-model [36, 45] the traffic at some time $t \in]0, T[$ and location $x \in \mathbb{R}$ is expressed by means of a traffic density $\rho(t, x) \in [0, \rho_{\max}]$. Moreover, the averaged velocity at some point in space and time is assumed to only depend on the current traffic density at this point, i. e. $v = v(\rho)$. The evolution of the traffic distribution is then described by

$$\rho_t + \hat{f}(\rho)_x = 0, \qquad \hat{f}(\rho) := \rho v(\rho).$$

There are various suggestions for the velocity function $\rho \mapsto v(\rho)$ in the literature that lead often to a strictly concave flux function $\hat{f}$. For example, the Greenshield velocity for

$n = 1$ given by $v(\rho) := v_{\max}\,(1 - \frac{\rho}{\rho_{\max}})$ yields a strictly concave flux function $\hat{f}$. A simple transformation of variables,

$$y := 1 - \frac{\rho}{\rho_{\max}}, \qquad f(y) := -\hat{f}((1-y)\rho_{\max}\,) = (y-1)\rho_{\max}\,v((1-y)\rho_{\max}\,),$$

makes the traffic flow model fit into our setting. The possibility of adding a traffic light to the above model was already discussed in the original works [36, 45] and by many authors afterwards. In the terms of the traffic light problem (2.5) an off-switching means switching the traffic light from green to red. Consequently, an on-switching means switching back from red to green.

3. General and structural properties of the on/off-switching problem

In this section we analyze the structure of solutions to on/off-switching problems. From this point we focus on the switching time controls and consider fixed initial data, i.e. we choose a fixed function

$$u_I \in \mathrm{PC}^1(\mathbb{R}; x_1, \ldots, x_{n_x}), \quad 0 \le u_I \le 1. \tag{3.1}$$

By definition a solution of an OOSP is a concatenation of solutions to a finite number of IVPs and IBVPs. Therefore, the existence, uniqueness and stability properties provided in the literature carry over to the present problem.

We will work under the following assumptions.

(A1) The flux function satisfies $f \in \mathrm{C}^2_{\mathrm{loc}}(\mathbb{R})$, $f(0) = f(1) = 0$ and there exists $m_{f''} > 0$ such that $f'' \ge m_{f''}$. The source term satisfies $g \in \mathrm{C}\left([0,T]; \mathrm{C}^1(\mathbb{R} \times [0,1])\right)$ and for all $(t,x) \in [0,T] \times \mathbb{R}$

$$g(t,x,y) \ge 0 \text{ for all } y \le 0, \qquad g(t,x,y) \le 0 \text{ for all } y \ge 1 \tag{3.2}$$

holds. Finally there is $\varepsilon_g > 0$ such that $g(t,\cdot,y)|_{[-\varepsilon_g,\varepsilon_g]} = 0$.

Under this assumption one can show that (2.5) is a well posed problem.

Corollary 3.1 (Existence and uniqueness for on/off-switching problems) *Let* (A1) *hold and consider* u_I *as in* (3.1). *Then for every* $\sigma \in \Sigma$ *there exists a unique entropy solution* $y = y(\sigma) \in \mathrm{L}^\infty(\Omega_T)$ *of* (2.5). *After a possible modification on a set of measure zero it even holds* $y \in \mathrm{C}([0,T]; \mathrm{L}^1_{\mathrm{loc}}(\mathbb{R}))$ *and* $y(t,\cdot) \in \mathrm{BV}_{\mathrm{loc}}(\mathbb{R})$ *for all* $t \in [0,T]$. *The solution satifies* $y(t,x) \in [0,1]$ *for almost all* $(t,x) \in \Omega_T$.

Moreover, there is a constant $L_\Sigma > 0$ *such that for all* $\tilde{\sigma}, \hat{\sigma} \in \Sigma$ *and all* $t \in [0,T]$ *holds*

$$\|y(t,\cdot;\tilde{\sigma}) - y(t,\cdot;\hat{\sigma})\|_{1,\mathrm{loc}} \le L_\Sigma \,\|\tilde{\sigma} - \hat{\sigma}\|\,.$$

Proof. The existence of a unique solution is a direct consequence of the respective theorems for Cauchy and initial-boundary value problems, see for example [18, 33, 34, 38, 50], and the fact, that y is a concatenation of solutions to such problems. The same holds for the regularity of y.

We prove the L^1-stability. We define the componentwise maximum and minimum of $\hat{\sigma}$ and $\tilde{\sigma}$ as $\overline{\sigma}$ and $\underline{\sigma}$. By definition $\overline{\sigma}$ and $\underline{\sigma}$ are strictly monotone increasing. We will use the fact, that a solution y to a Cauchy problem can be interpreted as the solution of two initial-boundary value problems on $\mathbb{R}^\pm$ with boundary data $y(\cdot,0\pm)$ and that the difference between two solutions to (2.5) is uniformly bounded by 1.

We start with $\hat{t} = \overline{\sigma}_1 = \underline{\sigma}_1 = 0$ and iterate while $\hat{t} < T$:

(i) If $\hat{t} = \overline{\sigma}_i < \underline{\sigma}_{i+1}$ for some i, $\hat{t}$ is the lower endpoint of a time interval on which $y(\cdot;\tilde{\sigma})$ and $y(\cdot;\hat{\sigma})$ are in the same phase, i.e. both are in the $\lceil \frac{i}{2} \rceil$-th off-phase or on-phase. If i is odd, $\hat{t}$ is the beginning of an on-phase and we apply the L^1-stability for Cauchy problems on $[\overline{\sigma}_i, \underline{\sigma}_{i+1}] \times \mathbb{R}$ and obtain, that for every $t \in [\overline{\sigma}_i, \underline{\sigma}_{i+1}]$

$$\|y(t,\cdot;\tilde{\sigma}) - y(t,\cdot;\hat{\sigma})\|_{1,\mathrm{loc}} \leq L_C \|y(\overline{\sigma}_i,\cdot;\tilde{\sigma}) - y(\overline{\sigma}_i,\cdot;\hat{\sigma})\|_{1,\mathrm{loc}}$$

holds. If i is even, $\hat{t}$ is the beginning of an off-phase and we apply the L^1-stability for the two initial-boundary value problems on $[\overline{\sigma}_i, \underline{\sigma}_{i+1}] \times \mathbb{R}^\pm$ for identical boundary data and obtain for every $t \in [\overline{\sigma}_i, \underline{\sigma}_{i+1}]$ that

$$\|y(t,\cdot;\tilde{\sigma}) - y(t,\cdot;\hat{\sigma})\|_{1,\mathrm{loc}} \leq 2L_B \|y(\overline{\sigma}_i,\cdot;\tilde{\sigma}) - y(\overline{\sigma}_i,\cdot;\hat{\sigma})\|_{1,\mathrm{loc}}.$$

Afterwards we set $\hat{t} = \underline{\sigma}_{i+1}$ and reiterate.

(ii) If $\hat{t} = \underline{\sigma}_i < \overline{\sigma}_i$ for some i, we define j as the smallest $j \geq i$ such that $\overline{\sigma}_j < \underline{\sigma}_{j+1}$, i.e., condition (i) holds. We apply the L^1-stability for the two IBVPs on $[\underline{\sigma}_i, \overline{\sigma}_j] \times \mathbb{R}^\pm$ for different boundary data, i.e. for every $t \in [\underline{\sigma}_i, \overline{\sigma}_j]$

$$\|y(t,\cdot;\tilde{\sigma}) - y(t,\cdot;\hat{\sigma})\|_{1,\mathrm{loc}} \leq 2L_B \left(\|y(\underline{\sigma}_i,\cdot;\tilde{\sigma}) - y(\underline{\sigma}_i,\cdot;\hat{\sigma})\|_{1,\mathrm{loc}} + \overline{\sigma}_j - \underline{\sigma}_i \right).$$

From the definition of j we know that for every $k \in \{i,\dots,j-1\}$ the inequality $\underline{\sigma}_{k+1} \leq \overline{\sigma}_k$ holds. Thus, by simple estimation and using a telescope sum, we obtain $\overline{\sigma}_j - \underline{\sigma}_i \leq \sum_{k=i}^{j} (\overline{\sigma}_k - \underline{\sigma}_k) = \sum_{k=i}^{j} |\tilde{\sigma}_k - \hat{\sigma}_k|$. We set $\hat{t} = \overline{\sigma}_j$ and go to (i) with $i = j$.

Since the loop terminates after at most $4n_\sigma + 2$ iterations, we see that the assertion holds for $L_\Sigma := (\max\{2L_B, L_C\})^{4n_\sigma+2}$ and the $\|\cdot\|_1$-norm on the righthand side. ■

Using the regularity properties, we conclude that for an entropy solution $y \in \mathrm{L}^\infty(\Omega_T) \cap \mathrm{C}([0,T];\mathrm{L}^1_{\mathrm{loc}}(\mathbb{R}))$ and all $(t,x) \in (0,T] \times \mathbb{R}$ the one-sided limits $y(t,x-)$ and $y(t,x+)$ exist. Whenever $x \neq 0$ or $(t,x) \in \Omega_{\mathrm{on},i}$, they satisfy $y(t,x-) \geq y(t,x+)$. We choose a pointwise defined representative of $y \in \mathrm{C}([0,T];\mathrm{L}^1_{\mathrm{loc}}(\mathbb{R}))$ and if $(t,x) \notin]\sigma^i_{\mathrm{off}}, \sigma^i_{\mathrm{on}}] \times \{0\}$, identify $y(t,x)$ with one of the limits $y(t,x-)$ or $y(t,x+)$.

We recall the definition of generalized characteristics in the sense of [19].

Definition 3.2 (Generalized characteristics) A Lipschitz curve

$$[\alpha,\beta] \subset [0,T] \to \Omega_T, \; t \mapsto (t,\xi(t))$$

is called a *generalized characteristic* on $[a,b]$ if

$$\dot{\xi}(t) \in [f'(y(t,\xi(t)+)), f'(y(t,\xi(t)-))], \quad \text{a.e. on } [\alpha,\beta]. \tag{3.3}$$

The generalized characteristic is called *genuine* if the lower and upper bound in (3.3) coincide for almost all $t \in [\alpha,\beta]$.

In the following we will also call ξ a (generalized) characteristic instead of $t \mapsto (t,\xi(t))$. It will also be useful to introduce notions of *extreme* or *maximal/minimal characteristics*

$\xi_\pm$, that satisfy

$$\dot\xi_\pm(t) = f'(y(t,\xi(t)\pm)) \quad \text{for a.a. } t.$$

Assumption (A1) yields, that y is bounded in $L^\infty(\Omega_T)$ and hence the maximum speed of a generalized characteristic is bounded, too. Therefore, characteristics either exist for the whole time period $[0,T]$ or meet the artificial boundary created by the off-mode condition at $x=0$ at some point $(\theta, 0\pm) \in \Omega^j_{\text{off},i}$, $i \in \{1,\dots,n_\sigma\}$, $j=1,2$. Moreover it can be shown [19] that (3.3) can be restricted to

$$\dot\xi(t) = \begin{cases} f'(y(t,\xi(t))) & \text{if } f'(y(t,\xi(t)+)) = f'(y(t,\xi(t)-)), \\ \frac{[f(y(t,\xi(t)))]}{[y(t,\xi(t))]} & \text{if } f'(y(t,\xi(t)+)) \neq f'(y(t,\xi(t)-)), \end{cases} \quad \text{a.e. on } [\alpha,\beta],$$

where for $\varphi \in \mathrm{BV}(\mathbb{R})$ the expression

$$[\varphi(x)] := \varphi(x-) - \varphi(x+)$$

denotes the jump of φ at x.

The next proposition collects useful properties for the solution y of (2.5) along generalized characteristics that do not touch the end points $(\sigma^i_{\text{off}}, 0)$, $(\sigma^i_{\text{on}}, 0)$ of an off-phase.

Proposition 3.3 (Structure of BV-solutions to the OOSP) *Let assumption* (A1) *hold. Consider an entropy solution y of the on/off-switching problem* (2.5) *for $\sigma \in \Sigma$, see* (2.4)*, and $u_I \in \mathrm{BV}_{\text{loc}}(\Omega;[0,1])$.*

Let ξ be a generalized characteristic on Ω, defined on a maximal interval $]\alpha,\beta[$ in $]0,\bar t] \subseteq]0,T[$. Then the following holds true:

(i) If ξ is an extreme backward characteristic, i. e., $\xi = \xi_\pm$, then ξ is genuine, i. e., $y(t,\xi_\pm(t)-) = y(t,\xi_\pm(t)+)$ for almost all $t \in]\alpha,\beta[$.

(ii) If ξ is genuine, then it satisfies

$$\xi(t) = \zeta(t), \quad y(t,\xi(t)) = v(t), \qquad t \in]\alpha,\beta[,$$

where (ζ, v) is a solution of the characteristic equation

$$\dot\zeta(t) = f'(v(t)), \tag{3.4a}$$
$$\dot v(t) = g(t,\zeta(t),v(t)). \tag{3.4b}$$

In particular, two different genuine characteristics may intersect only at their end points. For extreme characteristics $\xi_\pm$ the initial values are given by

$$(\zeta,v)(\bar t) = (\bar x, y(\bar t,\bar x\pm)). \tag{3.4c}$$

(iii) If ξ is genuine and $\xi(\beta) := \lim_{t\nearrow\beta}\xi(t) \in \Omega$, then

$$\xi(\beta) = \zeta(\beta), \quad y(t,\xi(\beta)-) \geq v(\beta) \geq y(t,\xi(\beta)+).$$

(iv) If ξ is genuine, $\alpha = 0$ and $z := \xi(0) := \lim_{t\searrow 0}\xi(t) \in \Omega$, then

$$z = \zeta(0), \quad u_I(z-) \leq v(0) \leq u_I(z+).$$

(v) If there exists $i \in \{1, \dots, n_\sigma\}$, $\theta \in]\sigma^i_{\text{off}}, \sigma^i_{\text{on}}[$ satisfying $f'(y(\theta, 0+)) < 0$, then there exists a genuine characteristic ξ on $]\alpha, \theta[$ satisfying

$$\xi(\theta) := \lim_{t \nearrow \theta} \xi(t) = 0, \qquad \dot{\xi}(\theta) := \lim_{t \nearrow \theta} \dot{\xi}(t) = f'(y(\theta, 0+)).$$

(vi) If there exists $i \in \{1, \dots, n_\sigma\}$, $\theta \in]\sigma^i_{\text{off}}, \sigma^i_{\text{on}}[$ satisfying $f'(y(\theta, 0-)) > 0$, then there exists a genuine characteristic ξ on $]\alpha, \theta[$ satisfying

$$\xi(\theta) = 0, \qquad \dot{\xi}(\theta) = f'(y(\theta, 0-)).$$

Let in the following $\alpha \in]\sigma^i_{\text{off}}, \sigma^i_{\text{on}}[$ for some $i \in \{1, \dots, n_\sigma\}$

(vii) If ξ is genuine and $\xi(\alpha) = 0$, $\dot{\xi}(\alpha) > 0$, then we have $v(\alpha) = 1$.

(viii) If ξ is genuine and $\xi(\alpha) = 0$, $\dot{\xi}(\alpha) < 0$, then we have $v(\alpha) = 0$.

Let in the following $\beta \in]\sigma^i_{\text{off}}, \sigma^i_{\text{on}}[$ for some $i \in \{1, \dots, n_\sigma\}$

(ix) If ξ is genuine and $\xi(\beta) = 0$, $\dot{\xi}(\beta) < 0$, then we have $v(\beta) \leq 0$.

(x) If ξ is genuine and $\xi(\beta) = 0$, $\dot{\xi}(\beta) > 0$, then we have $v(\beta) \geq 1$.

Proof. The above proposition is a consequence of the application of the results of [19] and [40, §3] to the present problem (2.5).

Assertions (i)–(iv) describe the behaviour of characteristics away from the artificial boundary introduced by off-switchings and follow simply from the classical results by Dafermos [19] for the Cauchy problem.

The remaining statements describe the situation at the artificial boundary. The existence of backward characteristics emanating from a point at the boundary with outgoing characteristic speed in (v) and (vi) follows from [40, Lem. 4], where an initial-boundary value problem with constant boundary data is considered. We recall that the boundary data for the right boundary of the incoming arc is given by 0 and for the left boundary of the outoging arc by 1. From [40, Prop. 3.2] we obtain that the limit of the function y along backward characteristics ending at the artificial boundary is equal to the boundary data. This shows (vii) for the outgoing and (viii) for the incoming arc. Statements (ix) and (x) describe the situation for forward characteristics and are consequences of [40, Prop. 3.3]. ∎

The following lemma on the differentiability of the solution operator of the characteristic equation (3.4) is a consequence of a result on ordinary differential equations (cf. [47, Prop. 3.4.5, Lem. 3.4.6] or [44, §5.6]). Together with Proposition 3.3 this lemma can be used to show local differentiability properties of a solution y to the OOSP. The occurring derivatives can be expressed by means of the solution $(\delta\zeta, \delta v)(\cdot; \theta, z, w; \delta\theta, \delta z, \delta w)$ of the *linearized characteristic equation*

$$\dot{\delta\zeta}(t) = f''(v(t))\delta v(t) \tag{3.5a}$$

$$\dot{\delta v}(t) = g_x(t, \zeta(t), v(t))\delta\zeta(t) + g_w(t, \zeta(t), v(t))\delta v(t) \tag{3.5b}$$

$$(\delta\zeta, \delta v)(\theta) = (\delta z - f'(w)\delta\theta, \delta w - g(\theta, z, w)\delta\theta). \tag{3.5c}$$

Lemma 3.4 *Let* (A1) *hold and denote for every $(\theta, z, w) \in [0, T] \times \mathbb{R} \times \mathbb{R}$ by $(\zeta, v)(\cdot; \theta, z, w)$ the solution of* (3.4a)-(3.4b) *for initial data*

$$(\zeta, v)(\theta; \theta, z, w) = (z, w).$$

Let M_w be given and set $\mathcal{B} := [0,T] \times \mathbb{R} \times]-M_w, M_w[$, then the mapping

$$(\theta, z, w) \in \mathcal{B} \longmapsto (\zeta, v)(\cdot, \theta, z, w) \in \mathrm{C}([0,T])^2$$

is Lipschitz continuous and continuously Fréchet-differentiable and on $\mathcal{B}$ the right hand side is uniformly Lipschitz w.r.t. t. The derivative is given in terms of the solution of the linearized characteristic equation (3.5) by

$$\mathrm{d}_{(\theta,z,w)}(\zeta, w) \cdot (\delta\theta, \delta z, \delta w) = (\delta\zeta, \delta v)(\cdot; \theta, z, w; \delta\theta, \delta z, \delta w).$$

Finally, for any closed $S \subset [0,T] \times \mathbb{R}$, any fixed $(\bar{\theta}, \bar{z}) \in [0,T] \times \mathbb{R}$ and any bounded intervals $\mathcal{T} \subseteq [0,T]$, $\hat{I}$ the mappings

$$(\theta, u_B) \in \mathrm{C}(S; \mathcal{T}) \times \mathrm{C}^1(\mathcal{T}) \longmapsto (\zeta, v)(\cdot_t, \theta, \bar{z}, u_B(\theta)) \in \mathrm{C}(S)^2,$$
$$(z, u_0) \in \mathrm{C}(S; \hat{I}) \times \mathrm{C}^1(\hat{I}) \longmapsto (\zeta, v)(\cdot_t, \bar{\theta}, z, u_0(z)) \in \mathrm{C}(S)^2$$

are continuously Fréchet-differentiable. Here $\cdot_t$ denotes the t-part of a point $(t,x) \in S$.

Lemma 3.4 is a direct generalization of [47, Lem. 3.4.6] to the case where the dependence on the time θ where the initial datum is specified, is considered, too, and can be obtained by standard calculus. The interested reader is referred to [41, Lem. 3.1.15] for the proof of the extension.

We now investigate the solution $y = (y_1, y_2)$ during an off-phase and at the beginning of the next on-phase.

Lemma 3.5 *Let (A1) hold and u_I be as in (3.1). Denote by $y = (y_1, y_2)$ the solution to (2.5). Consider $i = 1, \ldots, n_\sigma$, $\varepsilon_g > 0$ from (A1) and*

$$M_{f'} := \max(-f'(0), f'(1)) > 0. \tag{3.6}$$

Then the following holds true:

(i) *Let $t^* \in [\sigma^i_{\mathrm{off}}, \sigma^i_{\mathrm{on}}[$ such that $y_1(t^*, 0-) < 1$. Then there exist $0 < \delta \le \varepsilon_g$ and $m_{\dot\eta} > 0$ such that $y_1(t,x) = 0$ holds for all $(t,x) \in]t^*, \sigma^i_{\mathrm{on}}] \times]-\delta, 0[$ with $x > -m_{\dot\eta}(t - t^*)$.*

(ii) *Let $t^* \in [\sigma^i_{\mathrm{off}}, \sigma^i_{\mathrm{on}}[$ such that $y_2(t^*, 0+) > 0$. Then there exist $0 < \delta \le \varepsilon_g$ and $m_{\dot\eta} > 0$ such that $y_2(t,x) = 1$ holds for all $(t,x) \in]t^*, \sigma^i_{\mathrm{on}}] \times]0, \delta[$ with $x < m_{\dot\eta}(t - t^*)$.*

(iii) *Let $\tilde{\varepsilon} \in]0, \varepsilon_g]$ be such that $y(\sigma^i_{\mathrm{off}}, \cdot)$ is bounded away from 0 and 1 on $[-\tilde{\varepsilon}, \tilde{\varepsilon}]$, then there are $0 < \delta \le \tilde{\varepsilon}$ and $m_{\dot\eta} > 0$ such that*

$$y_1(t,x) = 0, \qquad (t,x) \in D_{\mathrm{off}} \cap \Omega^1_{\mathrm{off},i}, \tag{3.7}$$
$$y_2(t,x) = 1, \qquad (t,x) \in D_{\mathrm{off}} \cap \Omega^2_{\mathrm{off},i} \tag{3.8}$$

holds with $D_{\mathrm{off}} := \left\{(t,x) \in (\sigma^i_{\mathrm{off}}, \sigma^i_{\mathrm{on}}] \times (-\delta, \delta) \;:\; |x| < m_{\dot\eta}\left(t - \sigma^i_{\mathrm{off}}\right)\right\}$.

(iv) *If there exists $\tilde{\delta} \in]0, \varepsilon_g]$ such that $y(\sigma^i_{\mathrm{on}}, \cdot) = \mathbb{1}_{I_2}$ holds on $[-\tilde{\delta}, \tilde{\delta}]$, then*

$$y(t,x) = \min\left(1, \max\left(f'^{-1}\left(\frac{x}{t - \sigma^i_{\mathrm{on}}}\right), 0\right)\right), \qquad (t,x) \in D_{\mathrm{on}} \tag{3.9}$$

holds with $D_{\mathrm{on}} := \left\{(t,x) \in]\sigma^i_{\mathrm{on}}, \sigma^{i+1}_{\mathrm{off}}[\times]-\tilde{\delta}, \tilde{\delta}[\;:\; |x| < \tilde{\delta} - M_{f'}(t - \sigma^i_{\mathrm{on}})\right\}$.

(v) If the conditions from (iv) *are satisfied for some* $\bar{\sigma}_{\text{on}}^i$*, then we can reduce* $\tilde{\delta}$ *such that with* $\tilde{\tau} := \tilde{\delta}/(4M_{f'}) > 0$ *and* $0 < \rho < \tilde{\tau}$ *small enough holds*

$$y(t, x; \tilde{\sigma}) = \min\left(1, \max\left(f'^{-1}\left(\frac{x}{t - \tilde{\sigma}_{\text{on}}^i}\right), 0\right)\right) \tag{3.10}$$

for all $\tilde{\sigma} \in B_\rho^\Sigma(\bar{\sigma}) := \{\sigma \in \Sigma \ : \ \|\sigma - \bar{\sigma}\|_\infty < \rho\}$ *and all* $(t, x) \in]\tilde{\sigma}_{\text{on}}^i, \bar{\sigma}_{\text{on}}^i + \tilde{\tau}[\times \mathbb{R}$ *such that*

$$x \in] - \frac{2}{3}\tilde{\delta} + f'(0)(t - \bar{\sigma}_{\text{on}}^i - \tilde{\tau}), \frac{2}{3}\tilde{\delta} + f'(1)(t - \bar{\sigma}_{\text{on}}^i - \tilde{\tau})[, \tag{3.11}$$

and the latter interval contains $] - \frac{1}{6}\tilde{\delta}, \frac{1}{6}\tilde{\delta}[$ *by the definition of* $\tilde{\tau}$ *and* ρ*.*

Proof. We recall that by the BLN-boundary condition for almost every $t \in]\sigma_{\text{off}}^i, \sigma_{\text{on}}^i[$ we have $y(t, 0\pm) \in \{0, 1\}$.

Consider the setting of (i). We set

$$\tilde{\varepsilon} := -\frac{1}{2} \cdot \inf\{x \in] - \varepsilon_g, 0[\ : \ y_1(t^*, x) < 1\} \quad \text{and} \quad \tau := \frac{\tilde{\varepsilon}}{2M_{f'}}.$$

If $y_1(t^*, \cdot)$ is constantly equal to 0 on $] - \tilde{\varepsilon}, 0[$, we choose $m_{\dot{\eta}} := f'(0)$, otherwise we set

$$m_{\dot{\eta}} := \inf_{x \in]-\tilde{\varepsilon}, 0[} \left| \frac{f(y_1(t^*, x)) - f(0)}{y_1(t^*, x)} \right|.$$

Consider the unique generalized forward characteristic η through $(t^*, 0)$. For every $t \in]t^*, t^* + \tau[$ the maximal backward characteristic through $(t, \eta(t))$ is a straight line ending at the artificial boundary at $x = 0$. Hence, by Proposition 3.3 (viii) we have $y_1(t, \eta(t)+) = 0$. Moreover, the minimal backward characteristic is a straight line, too, and intersects the line $\{t^*\} \times] - \tilde{\varepsilon}, 0]$. Hence, the right limit of $y_1(t, \cdot)$ in $\eta(t)$ is equal to 0 while the left limit is contained in $\{f(y_1(t^*, x)) \ : \ x \in] - \tilde{\varepsilon}, 0[\}$ and in particular bounded away from 1. From this we conclude that $\dot{\eta}(t) \leq -m_{\dot{\eta}} < 0$ holds for every $t \in]t^*, t^* + \tau[$.

Now let $\delta := \tau m_{\dot{\eta}}$ and $t_\delta := \inf\left(\{t \ : \ \eta(t) = -\delta\} \cup \{\sigma_{\text{on}}^i\}\right)$. Using similar arguments as before, we obtain, that y_1 is constantly equal to 0 on $\{(t, x) \in [t^*, t_\delta] \times [-\delta, 0[\ : \ \eta(t) < x\}$.

(i) can now be proven by induction. From the previous considerations we know that

$$y_1(\hat{t}, \cdot)\big|_{]-\delta, 0[} \equiv 0 \tag{3.12}$$

holds true for $\hat{t} = t_\delta$. We prove that, whenever (3.12) holds for some $\hat{t} \in]\sigma_{\text{off}}^i, \sigma_{\text{on}}^i[$, the same must be true for all $t \in [\hat{t}, \min(\hat{t} + \hat{\tau}, \sigma_{\text{on}}^i)]$, where

$$\hat{\tau} := \frac{|f'(0)|}{3\,\|f''\|_{C([0,1])}\,\|g\|_\infty} > 0.$$

To prove this claim, we show first that the unique forward characteristic through $(\hat{t}, -\delta)$, again denoted by η, satisfies $\eta(t) \leq -\delta$ for all $t \in]\hat{t}, \min(\hat{t} + 2\hat{\tau}, \sigma_{\text{on}}^i)[$: For this purpose, assume that $\inf\{t \in]\hat{t}, T[\ : \ \eta(t) > -\delta\} < \min(\hat{t} + 2\hat{\tau}, \sigma_{\text{on}}^i)$ holds true. Then there exists $\tilde{t} \in]\hat{t}, \min(\hat{t} + 2\hat{\tau}, \sigma_{\text{on}}^i)[$ such that $\eta(\tilde{t}) > -\delta$ and $\dot{\eta}(\tilde{t}) > 0$. The maximal backward characteristic ξ^+ through $(\tilde{t}, \eta(\tilde{t}))$ must end at the artificial boundary at some time $\tilde{\theta}$. The curve $t \mapsto \xi^+(t)$ is a straight line—at least for t sufficiently close to $\tilde{t}$ or $\tilde{\theta}$, respectively. Since $\dot{\eta}(\tilde{t}) > 0$ we can deduce $y(\tilde{t}, \eta(\tilde{t})+) > 0$ and hence, the characteristic must not be a straight line for the whole interval $]\tilde{t}, \tilde{\tau}[$ but must leave and re-enter $]-\varepsilon_g, 0[\supset]-\delta, 0[$. From the characteristic equation (3.4) and Proposition 3.3 we obtain that the acceleration of ξ^+, i.e. $\ddot{\xi}^+$, is uniformly bounded by $\|f''\|_{\mathrm{C}([0,1])}\|g\|_\infty$ and hence the time between leaving and re-entering $]-\delta, 0[$ with speed $f'(0)$ is at least $3\hat{\tau}$. Thus, ξ^+ has to intersect η which is a contradiction. By using the same arguments we deduce, that $y(t, \cdot)|_{]-\delta,0[} \equiv 0$ must hold for all $t \in [\hat{t}, \min(\hat{t} + \hat{\tau}, \sigma^i_{\text{on}})]$.

This concludes the proof of (i). The proof of assertion (ii) is analogous and statement (iii) is a direct consequence of the first two.

For the proof of (iv) we consider the Cauchy problem on $\Omega_{\text{on},i}$. By the boundedness of speed of characteristics by $M_{f'}$ from (3.6) every backward characteristic through a point in D_{on} ends at $t = \sigma^i_{\text{on}}$ in $(-\delta, \delta)$. Hence, on D_{on} the solution y coincides with the solution of the Riemann problem for initial data $\mathbb{1}_{\mathbb{R}^+}$. That solution contains a rarefaction wave and is given by (3.9).

To prove (v) we reduce $\tilde{\delta} > 0$ obtained from (iv) such that there exists $\rho > 0$ with

$$\min\left(-f'(0), f'(1)\right)(\bar{\sigma}^i_{\text{on}} - \bar{\sigma}^i_{\text{off}} - \rho) > \tilde{\delta}.$$

We reduce $\rho > 0$ such that $\tilde{\tau} := \frac{\tilde{\delta}}{4M_{f'}} > \rho$ and find $\varepsilon > 0$ satisfying $\min(-f'(\varepsilon), f'(1-\varepsilon))(\bar{\sigma}^i_{\text{on}} + \tilde{\tau} - \bar{\sigma}^i_{\text{off}} - \rho) > \tilde{\delta}$. We use the local L^1-stability of the solution operator to (2.5) obtained in Corollary 3.1 to further reduce $\rho > 0$ such that

$$\left\|y(\bar{\sigma}^i_{\text{on}} + \tilde{\tau}, \cdot; \bar{\sigma}) - y(\bar{\sigma}^i_{\text{on}} + \tilde{\tau}, \cdot; \tilde{\sigma})\right\|_{1,]-\frac{3}{4}\tilde{\delta},\frac{3}{4}\tilde{\delta}[\setminus[-\frac{2}{3}\tilde{\delta},\frac{2}{3}\tilde{\delta}]} < \frac{\varepsilon\tilde{\delta}}{12}, \qquad \text{for all } \tilde{\sigma} \in B^\Sigma_\rho(\bar{\sigma}).$$

By the choice of $\tilde{\delta}$ and $\tilde{\tau}$, we know that $y(\bar{\sigma}^i_{\text{on}} + \tilde{\tau}, \cdot; \bar{\sigma})|_{]-\frac{3}{4}\tilde{\delta},\frac{3}{4}\tilde{\delta}[\setminus[-\frac{2}{3}\tilde{\delta},\frac{2}{3}\tilde{\delta}]} = \mathbb{1}_{\mathbb{R}^+}$. From the above inequality we deduce that for every $\tilde{\sigma} \in B^\Sigma_\rho(\bar{\sigma})$ there exist continuity points $x_l \in]-\frac{3}{4}\tilde{\delta}, -\frac{2}{3}\tilde{\delta}[$ and $x_r \in]\frac{2}{3}\tilde{\delta}, \frac{3}{4}\tilde{\delta}[$ of $y(\bar{\sigma}^i_{\text{on}} + \tilde{\tau}, \cdot; \tilde{\sigma})$ such that

$$y(\bar{\sigma}^i_{\text{on}} + \tilde{\tau}, x_l; \tilde{\sigma}) \in [0, \varepsilon[, \qquad y(\bar{\sigma}^i_{\text{on}} + \tilde{\tau}, x_r; \tilde{\sigma}) \in [1-\varepsilon, 1[.$$

Since in the considered area the source term vanishes, we conclude that the unique genuine backward characteristics $\xi_{l/r}$ through $(\bar{\sigma}^i_{\text{on}} + \tilde{\tau}, x_{l/r})$ are straight lines, traveling with speed $\dot{\xi}_l \in [f'(0), f'(\varepsilon)]$, $\dot{\xi}_r \in [f'(1-\varepsilon), f'(1)]$. By the choice of ε, the curves $\xi_{l/r}$ intersect $\{x = 0\}$ at some time $\theta_{l/r} > \bar{\sigma}^i_{\text{off}} + \rho \geq \tilde{\sigma}^i_{\text{off}}$. Moreover, the choice of $\tilde{\tau} > \rho > 0$ implies $\theta_{l/r} < \bar{\sigma}^i_{\text{on}} - \rho \leq \tilde{\sigma}^i_{\text{on}}$. By Proposition 3.3 the solution $y(\cdot; \tilde{\sigma})$ along ξ_l must be equal to 0 and along ξ_r it must be equal to 1. By the non-intersection property of genuine characteristics the same holds for all points $(t, x) \in]\tilde{\sigma}^i_{\text{on}}, \bar{\sigma}^i_{\text{on}} + \tilde{\tau}[\times \mathbb{R}$ such that

$$x \text{ satisfies (3.11) and } x \notin [f'(0)(t - \tilde{\sigma}^i_{\text{on}}), f'(1)(t - \tilde{\sigma}^i_{\text{on}})].$$

For $x \in [f'(0)(t - \tilde{\sigma}^i_{\text{on}}), f'(1)(t - \tilde{\sigma}^i_{\text{on}})]$ the backward characteristics must end in $(\tilde{\sigma}^i_{\text{on}}, 0)$. Hence, the final claim is proven. ■

4. Shift-differentiability

In this section we give the main result of this paper, that is the shift-differentiability of the control-to-state mapping for (2.5).

Before we introduce the notion of shift-variations and finally state the main theorems, we give a simple example to illustrate how shocks in the entropy solution cause the nondifferentiability of the control-to-state mapping.

Example 4.1 Let $y(\cdot;\sigma)$ be the solution of (2.5) for $T=3$, $n_\sigma=1$, $f(w)=\frac{1}{2}w(w-1)$, $g\equiv 0$ and $u_I\equiv 0$. Consider switching vectors of the form $\sigma=(0,\sigma_{\text{off}},2,3)$ with a single control variable $\sigma_{\text{off}}\in\,]0,1[$. Then a representative of the entropy solution is given by

$$y(t,x;(0,\sigma_{\text{off}},2,3))=\begin{cases}0 & \text{if } x\in\left]-\frac{t-\sigma_{\text{off}}}{4},\min\left(-\frac{t-2}{2},0\right)\right[,\\ 1 & \text{if } x\in\left]\max\left(\frac{t-2}{2},0\right),\frac{t-\sigma_{\text{off}}}{4}\right[,\\ \frac{x}{t-2}+\frac{1}{2} & \text{if } x\in\left]-\frac{t-2}{2},\frac{t-2}{2}\right[,\\ \frac{1}{2} & \text{else.}\end{cases}$$

For $\bar{t}\in\,]2,3[$ we consider the mapping $S:\,]0,1[\,\to \mathrm{L}^1(]-1,1[)$, $\sigma_{\text{off}}\mapsto y(\bar{t},\cdot;(0,\sigma_{\text{off}},2,3))$. Clearly, S is not differentiable, since the obvious candidate for the derivative, $\frac{1}{8}\delta_z-\frac{1}{8}\delta_{-z}$ with the Dirac measure δ_z at $z=\frac{t-\sigma_{\text{off}}}{4}$, does not belong to $\mathcal{L}(]0,1[,\mathrm{L}^1(]-1,1[))$. In fact, differentiability does only hold in the weak topology of the measure space $\mathcal{M}(]-1,1[)$.

4.1 *Definitions and preliminary work*

As we have seen in Example 4.1, the nondifferentiability of the solution operator is created by the shock discontinuity that changes its position depending on the control. In [11] and [47, 48] the authors used the specific structure to develop a suitable variational calculus. In this approach, the additive variations (e.g. in L^1) are augmented by possible horizontal shifts of discontinuities.

We recall the definition of the notions of shift-variations and shift-differentiability.

Definition 4.2 (Shift-variations, shift-differentiability)

(i) Let $a<b$ and $v\in \mathrm{BV}(a,b)$. For $a<x_1<x_2<\cdots<x_{n_x}<b$ we associate with $(\delta v,\delta x)$ the *shift-variation* $S_v^{(x_i)}(\delta v,\delta x)\in \mathrm{L}^1(a,b)$ of v by

$$S_v^{(x_i)}(\delta v,\delta x)(x):=\delta v(x)+\sum_{i=1}^{n}[v(x_i)]\operatorname{sgn}(\delta x_i)\mathbb{1}_{I(x_i,x_i+\delta x_i)}(x),$$

where $[v(x_i)]:=v(x_i-)-v(x_i+)$ and $I(\alpha,\beta):=[\min(\alpha,\beta),\max(\alpha,\beta)]$.

(ii) Let U be a real Banach space and $D\subset U$ open. Consider a locally bounded mapping $v:D\to \mathrm{L}^\infty(\mathbb{R})$, $u\mapsto v(u)$. For $\bar{u}\in U$ with $v(\bar{u})\in \mathrm{BV}(a,b)$, we call v *shift-differentiable at* $\bar{u}$ if there exist $a<x_1<x_2<\cdots<x_{n_x}<b$ and $D_s v(\bar{u})\in \mathcal{L}(U,\mathrm{L}^r(a,b)\times\mathbb{R}^{n_x})$ for some $r\in(1,\infty]$, such that for $\delta u\in U$, $(\delta v,\delta x):=D_s v(\bar{u})\cdot\delta u$ holds

$$\left\|v(\bar{u}+\delta u)-v(\bar{u})-S_v^{(x_i)}(\delta v,\delta x)\right\|_{1,]a,b[}=o(\|\delta u\|_U).$$

The mapping v is said to be continuously shift-differentiable at $\bar{u}$ if v is shift-differentiable in a neighborhood of $\bar{u}$ and if $D_s v(\cdot)$ and $x_i(\cdot)$, $v(\cdot)(x_i(\cdot)\pm)$, $i = 1, \ldots, n_x$ are continuous in $\bar{u}$.

As shown in [48, Lem. 2.3] this variational concept is strong enough to imply the Fréchet-differentiability of tracking type functionals as in (1.2) as long as y_d and $y(\bar{t}, \cdot)$ do not share discontinuities on $[a, b]$. The derivative is given by

$$\mathrm{d}_u J(y(u)) \cdot \delta u = (\psi_y(y(\bar{t}, \cdot; u), y_d), \delta y)_{2,[a,b]} + \sum_{i=1}^{n_x} \bar{\psi}_y(x_i)[y(\bar{t}, \cdot; u)]\delta x_i, \tag{4.1}$$

with

$$\bar{\psi}_y(x) := \int_0^1 \psi_y\left(y(\bar{t}, x+; u)) + \tau[y(\bar{t}, x; u)], y_d(x)\right) \mathrm{d}\tau. \tag{4.2}$$

Here, we use again the convention, that $y_d(x)$ is identified with one of the limits $y_d(x-)$ or $y_d(x+)$, which makes $\bar{\psi}_y$ be defined pointwise everywhere.

If the sets of discontinuities of $y(\bar{t}, \cdot; u)$ and y_d intersect, the reduced objective functional is still directionally differentiable. The directional derivative is given by (4.1) with $y_d(x_i)$ replaced by the corresponding one-sided limit $y_d(x_i + 0 \cdot \mathrm{sgn}(\delta x_i))$. For a proof we refer to [48, Lem. 2.3].

The proof of Theorem 4.6 and the formula for the gradient of the reduced objective function in Theorem 4.8, which will be stated in §4.2 and are the main results of this paper, are based on an appropriately defined adjoint state. Formally the adjoint equation is given by

$$\begin{aligned} p_t + f'(y)p_x &= -g_y(\cdot, y)p, \quad \text{on } \Omega_{\bar{t}} := [0, \bar{t}] \times \mathbb{R}, \\ p(\bar{t}, \cdot) &= p^{\bar{t}}, \qquad\qquad\quad \text{on } \mathbb{R}. \end{aligned} \tag{4.3}$$

Since the state y is in general discontinuous, the coefficients in (4.3) are discontinuous, too. This makes the analysis of the linear transport equation more involved. Nevertheless, for $g \equiv 0$ and Lipschitz continuous end data $p^{\bar{t}}$, Bouchut and James [8] give a definition of a *reversible solution* for (4.3), which satisfies a crucial duality relation.

In [47, 49] it was shown that the reversible solution of (4.3) is exactly the solution along the generalized characteristics of y. Using this characterization, the notion could be extended to more general source terms g, including all source terms satisfying (A3) and discontinuous end data.

In the present case we only consider the adjoint state on the set

$$\breve{\Omega}_{\bar{t}} := (]0, \bar{t}] \times \mathbb{R}) \setminus \left(\bigcup_{i=1}^{n_\sigma} ([\sigma_{\text{off}}^i, \sigma_{\text{on}}^i] \times \{0\}) \cup \breve{\Xi} \right),$$

where $\breve{\Xi}$ denotes the set of points (t, x) lying on a backward characteristic through a point $(\sigma_{\text{off}}^i, 0)$.

Definition 4.3 (Adjoint state) Let $p^{\bar{t}}$ be a bounded function that is the pointwise everywhere limit of a sequence (w_n) in $\mathrm{C}^{0,1}(\mathbb{R})$, with (w_n) bounded in $\mathrm{C}(\mathbb{R}) \cap \mathrm{W}^{1,1}_{\text{loc}}(\mathbb{R})$. The adjoint state p associated to (4.3) is characterized by the requirement that for every generalized characteristic ξ of y through $(\bar{t},\bar{x}) \in \Omega_T$ the function

$$t \mapsto p^{\xi}(t) = p(t,\xi(t))$$

is the solution of the ordinary differential equation

$$\begin{aligned}\dot{p^{\xi}}(t) &= -g_y(t,\xi(t),y(t,\xi(t)))p^{\xi}(t), \quad (t,\xi(t)) \in \breve{\Omega}_{\bar{t}},\\ p^{\xi}(\bar{t}) &= p^{\bar{t}}(\bar{x}).\end{aligned}$$

4.2 *Main results*

We are now able to formulate our main results for the on/off-switching problem. We state the shift-differentiability of the control-to-state operator from which we can deduce the total differentiability of the reduced objective functional. Finally, we give a formula for the gradient of the reduced objective in terms of an adjoint state.

We will work under the following assumptions:

(A2) $\Sigma_{\mathrm{ad}} \subset \Sigma$ is a closed set in $[0,T]^{2(n_\sigma+1)}$, with Σ defined in (2.4).
(A3) Assumption (A1) holds and in addition the source term satisfies $g \in \mathrm{C}^1([0,T] \times \mathbb{R} \times [0,1])$ and is affine linear w. r. t. y.
(A4) The off-switching points σ^i_{off} are nondegenerated in the sense of the following Definition 4.4.

The affine linearity of g w. r. t. y implies, that the coefficient at right hand side of (4.3) is continuous in space, which is needed in order to apply the theory of reversible solutions, cf. [48, §7].

Definition 4.4 (Nondegeneracy of σ^i_{off}) An off-switching point σ^i_{off} is called *nondegenerated*, if $x = 0$ is a continuity point and no shock generation point of $y(\sigma^i_{\mathrm{off}},\cdot;\tilde{\sigma})$, where $\tilde{\sigma}$ denotes the truncated switching vector $\tilde{\sigma} := (\sigma^0_{\mathrm{on}},\sigma^1_{\mathrm{off}},\sigma^1_{\mathrm{on}},\ldots,\sigma^{i-1}_{\mathrm{off}},\sigma^{i-1}_{\mathrm{on}},T)$. Moreover, the unique backward characteristic through $(\sigma^i_{\mathrm{off}},0)$ does not intersect any of the points $(\sigma^j_{\mathrm{off}},0)$, $j = 1,\ldots,i-1$. In addition, there is $t^* \in]\sigma^i_{\mathrm{off}},\sigma^i_{\mathrm{on}}[$ such that assertions (i) and (ii) of Lemma 3.5 are applicable.

Remark 4.5 The existence of t^* such that (i) and (ii) of Lemma 3.5 hold means, that the shock generated by the off-switching moves away from the boundary $x = 0$ until the next on-switching occurs.

The following main theorem on the shift-differentiability of the control-to-state mapping for the OOSP assumes a nondegeneracy condition to hold for all shocks at observation time $\bar{t}$. The formulation of that condition is rather technical and thus, postponed to §5. Roughly speaking it ensures that the shock structure at $\bar{t}$ does not change for small perturbations of the control.

Theorem 4.6 (Shift-Differentiability for on/off-switching problems) *Let* $\bar{\sigma} = (\bar{\sigma}^0_{\mathrm{on}},\bar{\sigma}^1_{\mathrm{off}},\bar{\sigma}^1_{\mathrm{on}},\ldots,\bar{\sigma}^{n_\sigma}_{\mathrm{on}},\bar{\sigma}^{n_\sigma+1}_{\mathrm{off}}) \in \Sigma_{\mathrm{ad}}$ *and* u_I *as in* (3.1). *Let* (A3), (A4) *hold and for every* $\sigma \in \Sigma_{\mathrm{ad}}$ *denote by* $y = y(\sigma) \in \mathrm{L}^\infty(\Omega_T) \cap \mathrm{C}([0,T];\mathrm{L}^1_{\mathrm{loc}}(\mathbb{R}))$ *the solution of the on/off-switching problem* (2.5). *Let* $a < b$ *and* $\bar{t} \in]\bar{\sigma}^{n_\sigma}_{\mathrm{on}},\bar{\sigma}^{n_\sigma+1}_{\mathrm{off}}[$ *such that* $y(\bar{t},\cdot;\bar{\sigma})$ *has on* $[a,b]$ *no shock generation points and only a finite number of shocks at* $a < \bar{x}_1 < \cdots < \bar{x}_{\bar{N}} < b$, *that all are neither degenerated according to Definition 5.1 nor shock interaction points.*

Consider the mapping

$$\sigma \in \Sigma_{\mathrm{ad}} \longmapsto y\left(\bar t, \cdot; \sigma\right) \in \mathrm{L}^1(a,b). \tag{4.4}$$

Then the following holds:

(i) If none of the backward characteristics through continuity points $x \in [a,b]$ of $y(\bar t, \cdot; \bar\sigma)$ intersect $(\bar\sigma^i_{\mathrm{on}}, 0)$ for some i, then the mapping (4.4) *is continuously shift-differentiable on the ball $B^{\Sigma}_{\rho}(\bar\sigma)$ for sufficiently small $\rho > 0$. The shift-derivative satisfies $D_s y(\bar t, \cdot; \bar\sigma) \in \mathcal{L}(\Sigma_0, \mathrm{PC}([a,b]; \bar x_1, \cdots, \bar x_{\bar N}) \times \mathbb{R}^{\bar N})$, where $\Sigma_0 := \{\nu \in \mathbb{R}^{2(n_\sigma+1)} : \nu_1 = \nu_{2(n_\sigma+1)} = 0\}$.*

(ii) If there are continuity points $x \in [a,b]$ of $y(\bar t, \cdot; \bar\sigma)$ that intersect $(\bar\sigma^i_{\mathrm{on}}, 0)$ for some i, then the mapping (4.4) *is shift-differentiable in $\bar\sigma$. The shift-derivative satisfies $D_s y(\bar t, \cdot; \bar\sigma) \in \mathcal{L}(\Sigma_0, \mathrm{PC}([a,b]; \tilde x_1, \cdots, \tilde x_{\tilde N}) \times \mathbb{R}^{\tilde N})$, where $(\tilde x_j)$ is the set of shock points extended by all continuity points of $y(\bar t, \cdot; \bar\sigma)$ with backward characteristics through some $(\bar\sigma^i_{\mathrm{on}}, 0)$.*

If in addition $f \in \mathrm{C}^3$, $g \in \mathrm{C}\left([0,T]; \mathrm{C}^2(\mathbb{R} \times [0,1])\right)$, and $u_I \in \mathrm{PC}^2(\mathbb{R}; x_1, \ldots, x_{n_x})$ then the above assumption on shocks and shock generation points holds for almost all $\bar t \in]0,T[$.

Proof. The proof is given in section 5. The fact that at almost all $\bar t \in]0,T[$ the assumption on shocks and shock generation points hold can be shown as in [48, Thm. 3.8]. ■

It is important to emphasize that for the on/off-switching problem also off-switching times, i.e. rarefaction centers, may explicitly be shifted, whereas for the initial (-boundary) value problem in [42, 48] only the shifting of shock creating discontinuities was allowed. This is a particular challenge in the analysis of the on/off-switching problem OOSP. We will use the fact that for OOSPs the solution in a neighborhood of an on switching rarefaction centers is thoroughly known, cf. Lemma 3.5.

From [48, Lem. 2.3] we can deduce the total differentiability for reduced objective functionals.

Corollary 4.7 *Let the assumptions of Theorem 4.6* (ii) *hold and consider J defined as in* (1.2)*. Then the reduced functional*

$$\sigma \in \Sigma_{\mathrm{ad}} \longmapsto J(y(\sigma)) \tag{4.5}$$

is directionally differentiable in $\bar\sigma$. The directional derivative is given by (4.1) *with $y_d(x_i)$ replaced by the corresponding one-sided limit $y_d(x_i + 0 \cdot \mathrm{sgn}(\delta x_i))$.*

If even the assumptions of Theorem 4.6 (ii) *are satisfied and y_d is continuous in a small neighborhood of $\{\bar x_1, \ldots, \bar x_{\bar N}\}$, then* (4.5) *is continuously differentiable on $B^{\Sigma}_{\rho}(\bar\sigma)$ for $\rho > 0$ small enough with derivative given by* (4.1), (4.2)*.*

In the following theorem we give a representation of the gradient of the reduced objective function based on the appropriate notion of an adjoint state from Definition 4.3.

Theorem 4.8 *Let the assumptions of Corollary 4.7 hold and let the terminal data $p^{\bar t}$ in* (4.3) *$p^{\bar t}$ be given by $\bar\psi_y$ defined in* (4.2)*. Then there exists an adjoint state p according to Definition 4.3 as the reversible solution of the adjoint equation* (4.3)*.*

The derivative of the reduced functional $\sigma \in \Sigma_{\mathrm{ad}} \mapsto \hat J(\sigma) = J(y(\sigma))$ in $\bar\sigma$ and direction

$\delta\sigma \in \Sigma_0$ *is given by*

$$\hat{J}'(\bar{\sigma}) \cdot \delta\sigma = \sum_{i=1}^{n_\sigma} \left(p_{\text{off}}^i \cdot \delta\sigma_{\text{off}}^i + p_{\text{on}}^i \cdot \delta\sigma_{\text{on}}^i \right),$$

where

$$p_{\text{on}}^i := \int_{f'(-1)}^{f'(0)} \lim_{s \searrow \bar{\sigma}_{\text{on}}^i} p(s, (s - \bar{\sigma}_{\text{on}}^i) \cdot w) \cdot w \cdot \left(f'^{-1} \right)'(w) \, \mathrm{d}w,$$
$$p_{\text{off}}^i := \left(p(\bar{\sigma}_{\text{off}}^i, 0+) - p(\bar{\sigma}_{\text{off}}^i, 0-) \right) \cdot f(y(\bar{\sigma}_{\text{off}}^i -, 0; \bar{\sigma})).$$

Proof. The proof is given in section 5. ■

One may also consider the initial data as additional control. The derivatives w. r. t. the initial data are then given as for the Cauchy problem without on/off-switching, cf. [48].

5. Proof of the main results

In this section we prove the results of §4.2. For the whole section we will work in the setting of Theorem 4.6, i.e. we assume (A3), (A4) to hold.

5.1 *Classification of continuity and shock points*

In this subsection we classify the different types of continuity points and shock points $\bar{x}$ of $y(\bar{t}, \cdot; \sigma)$. To this purpose, we denote for a continuity point $\bar{x}$ of $y(\bar{t}, \cdot; \sigma)$ the unique backward characteristic by $\bar{\xi}$. By Proposition 3.3 $\bar{\xi}$ is genuine and coincides with the solution $\zeta(\cdot; \bar{t}, \bar{x}, y(\bar{t}, \bar{x}; \sigma))$ of the characteristic equation (3.4). The genuine backward characteristic $\bar{\xi}$ propagates until it reaches the initial data, the boundary data during an off-phase or the rarefaction center $(\sigma_{\text{on}}^i, 0)$ at the beginning of an on-phase.

We start by collecting the classes of points where $\bar{\xi}$ does not touch the artificial boundary at $x = 0$ during an off-phase $[\sigma_{\text{off}}^i, \sigma_{\text{on}}^i]$ including the on-switching time. Thus, $\bar{\xi}$ exists on the whole interval $[0, \bar{t}]$ and ends in a point $\bar{z}$ at $t = 0$. Those points have already been categorized and analyzed in [48]. We briefly recall the classifications. We denote by $\bar{w} := v(0; \bar{t}, \bar{x}, y(\bar{t}, \bar{x}; \sigma))$ the value of the v-part of the solution of (3.4) corresponding to $\bar{\xi}$.

Case C**:** $\bar{z} \neq x_l$ *for* $l = 1, \ldots, n_x$.
There exists an interval J with $\bar{z} \in J$ and $u_I|_J \in \mathrm{C}^1(J)$ and

$$\frac{\mathrm{d}}{\mathrm{d}z} \zeta(t; 0, z, u_I(z))|_{z=\bar{z}} \geq 0, \quad t \in [0, \bar{t}]. \tag{5.1}$$

We say that $\bar{x}$ is of class C^c if even

$$\frac{\mathrm{d}}{\mathrm{d}z} \zeta(t; 0, z, u_I(z))|_{z=\bar{z}} \geq \beta > 0, \quad t \in [0, \bar{t}]. \tag{5.2}$$

As shown in [48] (5.2) holds if $(\bar{t}, \bar{x})$ is no shock generation point.

Case CB**:** *$\bar{z} = x_l$ for some $l \in \{1, \ldots, n_x\}$ and $u_I(x_l-) = u_I(x_l+)$.*
By the same arguments the one-sided derivatives satisfy (5.1). If even the one-sided version of (5.2) holds, we call $\bar{x}$ of case CB^c.

Case R**:** *$\bar{z} = x_l$ for some $l \in \{1, \ldots, n_x\}$ and $\bar{w} \in]u_I(x_l-), u_I(x_l+)[$.*
In this case we have

$$\frac{\mathrm{d}}{\mathrm{d}w}\zeta(t; 0, \bar{z}, w)|_{w=\bar{w}} \geq 0, \quad t \in [0, \bar{t}]. \tag{5.3}$$

The R^c-Case is characterized by the stronger inequality

$$\frac{\mathrm{d}}{\mathrm{d}w}\zeta(t; 0, \bar{z}, w)|_{w=\bar{w}} \geq \beta t > 0, \quad t \in]0, \bar{t}], \tag{5.4}$$

which again is ensured by the requirement that no shock is generated at $(\bar{t}, \bar{x})$.

Case RB**:** *$\bar{z} = x_l$ for some $l \in \{1, \ldots, n_x\}$, $u_I(x_l-) < u_I(x_l+)$ and $\bar{w} \in \{u_I(x_l+), u_I(x_l-)\}$.*
The point $(\bar{t}, \bar{x})$ lies on the left or right boundary of a rarefaction wave. The one-sided derivatives satisfy (5.1) and (5.3), respectively. If even (5.2) and (5.4) are satisfied, $\bar{x}$ is of class RB^c.

For brevity we collect these classes of continuity points with $\bar{\xi}$ starting from the initial data in the classes $X_I := \{C, CB, R, RB\}$ and $X_I^c := \{C^c, CB^c, R^c, RB^c\}$, respectively.

The following classes of continuity points $\bar{x}$ are special to the on/off-switching problem, namely points whose backward characteristics end at $\{x = 0\}$ during or at the end of an off-phase at some time $\bar{\theta} \in [\sigma_{\text{off}}^i, \sigma_{\text{on}}^i]$. We denote by $\bar{w} := v(\bar{\theta}; \bar{t}, \bar{x}, y(\bar{t}, \bar{x}; \sigma))$ the value of the v-part of the solution (ζ, v) of (3.4) corresponding to $\bar{\xi}$.

Case C_S**:** *$\bar{\theta} \in]\sigma_{\text{off}}^i, \sigma_{\text{on}}^i[$ for some $i \in \{1, \ldots, n_\sigma\}$.*
Depending on the orientation of $\bar{\xi}$ we have

$$\frac{\mathrm{d}}{\mathrm{d}\theta}\zeta(t; \theta, 0, 0)|_{\theta=\bar{\theta}} \geq 0, \qquad \bar{\theta} \leq t \leq \bar{t}, \qquad \text{if } \dot{\bar{\xi}}(\bar{\theta}+) < 0, \tag{5.5a}$$

$$\frac{\mathrm{d}}{\mathrm{d}\theta}\zeta(t; \theta, 0, 1)|_{\theta=\bar{\theta}} \leq 0, \qquad \bar{\theta} \leq t \leq \bar{t}, \qquad \text{if } \dot{\bar{\xi}}(\bar{\theta}+) > 0. \tag{5.5b}$$

We say that $\bar{x}$ is of class C_S^c if even

$$\frac{\mathrm{d}}{\mathrm{d}\theta}\zeta(t; \theta, 0, 0)|_{\theta=\bar{\theta}} \geq \beta > 0, \qquad \bar{\theta} \leq t \leq \bar{t}, \qquad \text{if } \dot{\bar{\xi}}(\bar{\theta}+) < 0, \tag{5.6a}$$

$$\frac{\mathrm{d}}{\mathrm{d}\theta}\zeta(t; \theta, 0, 1)|_{\theta=\bar{\theta}} \leq -\beta < 0, \qquad \bar{\theta} \leq t \leq \bar{t}, \qquad \text{if } \dot{\bar{\xi}}(\bar{\theta}+) > 0. \tag{5.6b}$$

By using the same arguments as in [48, §4.3], one can show that (5.6) holds if (5.5) is satisfied and $(\bar{t}, \bar{x})$ is no shock generation point.

Case R_S**:** *$\bar{\theta} = \sigma_{\text{on}}^i$ for some $i \in \{1, \ldots, n_\sigma\}$ and $\dot{\bar{\xi}}(\sigma_{\text{on}}^i+) = f'(\bar{w}) \in]f'(0), f'(1)[$.*
The point $\bar{x}$ is located in the interior of a rarefaction wave created by an on-switching. In this case we have

$$\frac{\mathrm{d}}{\mathrm{d}w}\zeta(t; \sigma_{\text{on}}^i, 0, w)|_{w=\bar{w}} \geq 0, \quad t \in [\sigma_{\text{on}}^i, \bar{t}]. \tag{5.7}$$

The R_S^c-Case is characterized by the stronger inequality

$$\frac{\mathrm{d}}{\mathrm{d}w}\zeta(t;\sigma_{\text{on}}^i,0,w)|_{w=\bar{w}} \geq \beta(t-\sigma_{\text{on}}^i) > 0, \quad t \in]\sigma_{\text{on}}^i, \bar{t}]. \tag{5.8}$$

which, again is ensured by the requirement that no shock is generated at $(\bar{t},\bar{x})$.

Case RB_S**:** *$\bar{\theta} = \sigma_{\text{on}}^i$ for some $i \in \{1,\ldots,n_\sigma\}$ and $\dot{\bar{\xi}}(\sigma_{\text{on}}^i+) = f'(\bar{w}) \in \{f'(0), f'(1)\}$.*
The point $\bar{x}$ is located at the boundary of a rarefaction wave created by an on-switching. The one-sided derivatives satisfy (5.7) and (5.5). If even (5.8) and (5.6) are satisfied, we say that $\bar{x}$ is of class RB_S^c.

Case CB_S**:** *$\bar{\theta} = \sigma_{\text{off}}^i$ for some $i \in \{1,\ldots,n_\sigma\}$.*
The genuine backward characteristic touches the i-th off-switching point with $\bar{w} \in \{0,1\}$.

If $\bar{w} = 0$, the characteristic $\bar{\xi}$ reaches the off-phase from the left. The right-sided derivative satisfies (5.5a) for $\bar{\theta} = \sigma_{\text{on}}^i$. Moreover, the characteristic does not end at this point but continues until $t < \sigma_{\text{on}}^i$ and either ends at the interior of an earlier off-phase $]\sigma_{\text{off}}^j, \sigma_{\text{on}}^j[$, $j < i$, in an earlier on-switching point σ_{on}^j, $j < i$, or at $t = 0$. Depending on its endpoint the left-sided derivative satisfies (5.5), (5.7), (5.1) or (5.3).

If $\bar{w} = 1$, the characteristic $\bar{\xi}$ reaches the off-phase from the right. The same properties as above hold with "left" and "right" swapped.

If the one-sided derivatives even satisfy the stronger inequalities (5.6), (5.8), (5.2) or (5.4), we say that $\bar{x}$ is of class CB_S^c.

The shock points are categorized by the classes of their minimal and maximal characteristics $\bar{\xi}_l$ and $\bar{\xi}_r$. We introduce the notion of a nondegenerated shock as follows.

DEFINITION 5.1 (Nondegeneracy of shock points) A point $\bar{x}$ of discontinuity of $y(\bar{t},\cdot;\sigma)$ is called nondegenerated, if it is no shock interaction point and is of class $X_l X_r$ with $X_l, X_r \in \{X_I^c, R_S^c, C_S^c\}$.

Using the introduced classifications, we are able to reformulate the nondegeneracy condition for off-switching points as follows.

Remark 5.2 An off-switching point σ_{off}^i is nondegenerated according to Definition 4.4 if and only if $x = 0$ is a continuity point of $y(\sigma_{\text{off}}^i, \cdot; (0, \sigma_{\text{off}}^1, \ldots, \sigma_{\text{on}}^{i-1}, T))$ of class $\check{X}^i \in \{X_I^c, R_S^c, RB_S^c, C_S^c\}$ and if there exists a time $t^* \in]\sigma_{\text{off}}^i, \sigma_{\text{on}}^i[$ such that assertions (i) and (ii) of Lemma 3.5 are applicable.

For nondegenerated off-switching points σ_{off}^i we will also refer to $\check{X}^i$ as the class of σ_{off}^i.

5.2 *Differentiability at continuity points*

As already mentioned, continuity points of class X_I^c have been considered in [48], where optimal control of the Cauchy problem has been studied. Since in the on-/off switching problem (2.5) the initial value and the source term are assumed to be fixed, the solution in continuity points of class X_I^c, for which the backward characteristic reaches the initial data, are independent of the control. Therefore, the results of [48] can be boiled down to the following corollary.

COROLLARY 5.3 *Let* (A3) *hold, consider u_I as in* (3.1)*, $\bar{\sigma} \in \Sigma$ and let $(\bar{t}, \bar{x})$ be a X_I^c-point. Then there exists a neighborhood S of the genuine backward characteristic $\bar{\xi}$ and*

$\rho > 0$ *such that for every* $s > 0$ *it holds that* $y(\cdot;\bar\sigma) \in \mathrm{C}^{0,1}(S \cap \{t \geq s\})$ *and*

$$y(t,x;\sigma) = y(t,x;\bar\sigma), \quad \forall\,(t,x) \in S, \quad \forall\,\sigma \in B^{\Sigma}_{\rho}(\rho).$$

We turn now to continuity points $(\bar t, \bar x)$ of class $C^{\mathrm{c}}_S, CB^{\mathrm{c}}_S, R^{\mathrm{c}}_S, RB^{\mathrm{c}}_S$, where the backward characteristic $\bar\xi$ ends at an artificial boundary $[\sigma^i_{\mathrm{off}}, \sigma^i_{\mathrm{on}}] \times \{\underline{0}\}$.

Let $\bar x$ be a continuity point of $y(\bar t,\cdot;\bar\sigma)$ of class C^{c}_S with $\bar\xi$ approaching the boundary at $(\bar\theta, 0)$ from the left, i.e. (5.6a) is satisfied. Then by continuity we can find $\theta_l < \bar\theta < \theta_r$ and $\kappa > 0$, such that (after a possible reduction of β) holds

$$\frac{\mathrm{d}}{\mathrm{d}\theta}\zeta(t;\theta,0,0) \geq \beta > 0, \quad \text{for all } (t,\theta) \in \mathbf{T}_{\bar t}, \tag{5.9}$$

where for every $s > 0$ the set $\mathbf{T}_s$ is defined by

$$\mathbf{T}_s := \{(t,\theta) \in [0,s] \times \mathcal{T} \;:\; t \geq \theta\} \quad \text{with} \quad \mathcal{T} =]\theta_l - \kappa, \theta_r + \kappa[\subset]\sigma^i_{\mathrm{off}}, \sigma^i_{\mathrm{on}}[.$$

Lemma 5.4 *Let* (A3) *hold and let* (5.9) *be satisfied for some* $\beta, \kappa > 0$*. Let* $\bar x$ *be a continuity point of* $y(\bar t,\cdot;\bar\sigma)$ *of class* C^{c}_S *with* $\bar\xi$ *approaching the boundary at* $(\bar\theta, 0)$ *from the left. Then the following holds true:*

(i) There exists $\tau > 0$ *such that*

$$\frac{\mathrm{d}}{\mathrm{d}\theta}\zeta(t;\theta,0,0) \geq \frac{\beta}{2} > 0, \qquad \forall (t,\theta) \in \mathbf{T}_{\bar t + \tau}.$$

(ii) Consider a point $(t,x) \in S = S(\tau)$*, where*

$$S(\tau) := \{(t,x) \in [\theta_l, \bar t + \tau] \times \mathbb{R} \;:\; x \in [\xi_l(t), \xi_r(\max(t,\theta_r))]\}$$

and $\xi_{l/r}(t) := \zeta(t;\theta_{l/r},0,0)$*. Then the equation*

$$x = \zeta(t;\theta,0,0)$$

is uniquely solvable w.r.t. θ *on* $\mathcal{T}$ *from* (5.9) *with solution* $\theta = \Theta(t,x)$*. Moreover, let* $Y_C(t,x)$ *be defined by*

$$Y_C(t,x) := v(t;\Theta(t,x),0,0).$$

Then

(iii) $\Theta, Y_C \in \mathrm{C}^1(S)$.

(iv) The mapping

$$x \in]\xi_l(t), \xi_r(\max(t,\theta_r))[\longmapsto (\Theta, Y_C)(t,x), \qquad t \in [\theta_l, \bar t + \tau[$$

is continuously differentiable with derivatives

$$\begin{aligned} \mathrm{d}_x\Theta(t,x) &= \left(\delta\zeta(t;\theta,0,0;1,0,0)\right)^{-1}, \\ \mathrm{d}_x Y_C(t,x) &= \delta v(t;\theta,0,0;1,0,0) \cdot \mathrm{d}_x\Theta(t,x) \end{aligned}$$

where $\theta = \Theta(t,x)$ *and* $(\delta\zeta, \delta v)$ *as defined in* (3.5).

Analogous results hold for continuity points $\bar{x}$ of $y(\bar{t},\cdot;\bar{\sigma})$ of class C_S^c with $\bar{\xi}$ approaching the boundary at $(\bar{\theta},0)$ from the right.

Proof. Lemma 5.4 is a special case of [42, Lem. 4.2] for constant boundary data and fixed source term. The assertions can be proven in the same way as [48, Lem. 5.1], see also the proof of Lemma 5.6 below. ■

Lemma 5.5 *Let* (A3) *hold, consider u_I as in* (3.1), *$\bar{\sigma} \in \Sigma$ and let $\bar{x}$ be a C_S^c-point of $y(\bar{t},\cdot;\bar{\sigma})$ with $\bar{\xi}$ approaching $x=0$ from the left. Then the following holds.*

(i) There exists a maximal open interval $I \ni \bar{x}$, such that $\{\bar{t}\} \times I$ contains no point of the shock set and that all backward characteristics through a point $(\bar{t},x) \in \{\bar{t}\} \times I$ intersect $\{x=0\}$ in a point $\theta \in]\bar{\sigma}_{\text{off}}^i, \bar{\sigma}_{\text{on}}^i[$ from the left. In particular, none of those characteristics intersects $\{x=0\}$ during another off-phase $[\bar{\sigma}_{\text{off}}^s, \bar{\sigma}_{\text{on}}^s]$ with $s \geq i+1$.

(ii) $y(\bar{t},\cdot;\bar{\sigma})$ is continuously differentiable on I.

(iii) Let $\hat{I} :=]x_l, x_r[$ be an interval with $x_l, x_r \in I$. Denote by $\xi_{l/r}$ the genuine backward characteristics through $(\bar{t}, x_{l/r})$ with endpoints $\theta_{l/r}$ at $\{x=0\}$. Then there exist $\kappa, \beta > 0$, such that (5.9) *is satisfied.*

(iv) After the possible reduction of τ from Lemma 5.4 there exists $\rho > 0$ such that

$$y(t,x;\sigma) = Y_C(t,x) \quad \forall (t,x) \in S,\ \forall \sigma \in B_\rho^\Sigma(\bar{\sigma}).$$

Analogous results hold for continuity points $\bar{x}$ of class C_S^c with $\bar{\xi}$ approaching the boundary at $(\bar{\theta},0)$ from the right.

Proof. Lemma 5.5 is a special case of [42, Lem. 4.3] for constant boundary data and fixed source term. The assertions can be proven in the same way as [48, Lem. 5.5]. ■

Now let $\bar{x}$ be a continuity point of $y(\bar{t},\cdot;\bar{\sigma})$ of class R_S^c with $\bar{\xi}$ approaching the rarefaction center $(\bar{\sigma}_{\text{on}}^i, 0)$, i.e. (5.8) is satisfied for $\sigma_{\text{on}}^i = \bar{\sigma}_{\text{on}}^i$. Then by continuity we can find $w_l < \bar{w} < w_r$ and $\kappa > 0$, such that (after a possible reduction of β) holds

$$\frac{\mathrm{d}}{\mathrm{d}w}\zeta(t;\bar{\sigma}_{\text{on}}^i,0,w) \geq \beta(t-\bar{\sigma}_{\text{on}}^i) > 0, \quad \forall\,(t,w) \in]\bar{\sigma}_{\text{on}}^i, \bar{t}] \times J_w, \tag{5.10}$$

where $J_w :=]w_l - \kappa, w_r + \kappa[$.

Lemma 5.6 *Let* (A3) *hold and let* (5.10) *be satisfied for some $\beta, \kappa > 0$. Then for every $\hat{\tau} \in]0, \frac{\varepsilon_g}{2\|f'\|_{\infty,J_w}}[$ the following holds true:*

(i) There exists $\tau > 0$ and $\rho > 0$ such that for all $\sigma \in B_\rho^\Sigma(\bar{\sigma})$ we have

$$\frac{\mathrm{d}}{\mathrm{d}w}\zeta(t;\sigma_{\text{on}}^i,0,w) \geq \frac{\beta}{2}(t-\sigma_{\text{on}}^i) > 0, \quad \forall\,(t,w) \in]\sigma_{\text{on}}^i, \bar{t}+\tau] \times J_w.$$

(ii) There exists $\rho \in]0,\hat{\tau}[$ such that for $\sigma \in B_\rho^\Sigma(\bar{\sigma})$ and $S := S(\tau,\sigma) := S_1(\tau) \cup S_2(\sigma)$, where

$$S_1 = S_1(\tau) := \{(t,x) \in [\bar{\sigma}_{\text{on}}^i + \hat{\tau}, \bar{t}+\tau] \times \mathbb{R} \ :\ x \in [\xi_l(t), \xi_r(t)]\},$$
$$S_2 = S_2(\sigma) := \left\{(t,x) \in]\sigma_{\text{on}}^i, \bar{\sigma}_{\text{on}}^i + \hat{\tau}] \times \mathbb{R} \ :\ x \in \left[z_l \frac{t-\sigma_{\text{on}}^i}{\hat{\tau}}, z_r \frac{t-\sigma_{\text{on}}^i}{\hat{\tau}}\right]\right\}$$

with $\xi_{l/r}(t) := \zeta(t;\bar\sigma^i_{\rm on},0,w_{l/r})$, $z_{l/r} := \xi_{l/r}(\bar\sigma^i_{\rm on}+\hat\tau)$ *for every* $(t,x)\in S$ *the equation*

$$x = \zeta(t;\sigma^i_{\rm on},0,w)$$

is uniquely solvable w.r.t. w *on* J_w *from* (5.10) *with solution* $w = W(t,x,\sigma)$. *Moreover, let* $Y_R(t,x,\sigma)$ *be defined by*

$$Y_R(t,x,\sigma) := v(t;\sigma^i_{\rm on},0,W(t,x,\sigma)).$$

Then he have:

(iii) For every $\sigma\in B^\Sigma_\rho(\bar\sigma)$ *and* $s>\sigma^i_{\rm on}$ *there holds* $W(\cdot,\sigma), Y_R(\cdot,\sigma)\in {\rm C}^1(S\cap\{t\ge s\})$.

(iv) The mapping

$$(x,\sigma)\in\,]\xi_l(t),\xi_r(t)[\,\times B^\Sigma_\rho(\bar\sigma)\longmapsto (W,Y_R)(t,x,\sigma),\qquad t\in[\bar\sigma^i_{\rm on}+\hat\tau,\bar t+\tau)$$

is continuously differentiable with derivatives

$$\begin{aligned}
{\rm d}_{(x,\sigma)}W(t,x,\sigma)\cdot(\delta x,\delta\sigma) &= \frac{\delta x - \delta\zeta(t;\sigma^i_{\rm on},0,w;\delta\sigma^i_{\rm on},0,0)}{\delta\zeta(t;\sigma^i_{\rm on},0,w;0,0,1)},\\
{\rm d}_{(x,\sigma)}Y_R(t,x,\sigma)\cdot(\delta x,\delta\sigma) &= \delta v(t;\sigma^i_{\rm on},0,w;\delta\sigma^i_{\rm on},0,0)\\
&\quad + \delta v(t;\sigma^i_{\rm on},0,w;0,0,1)\cdot {\rm d}_{(x,\sigma)}W(t,x,\sigma)\cdot(\delta x,\delta\sigma),
\end{aligned}$$

with $w = W(t,x,\sigma)$ *and* $(\delta\zeta,\delta v)$ *as defined in* (3.5).

(v) The mapping

$$\sigma\in B^\Sigma_\rho(\bar\sigma)\longmapsto (W,Y_R)(\cdot,\sigma)\in {\rm C}(S_1)$$

is continuously Fréchet-differentiable with derivative

$${\rm d}_\sigma(W,Y_R)(\cdot,\sigma)\cdot\delta\sigma = {\rm d}_{(x,\sigma)}(W,Y_R)(\cdot,\sigma)\cdot(0,\delta\sigma).$$

Proof. For (i) we consider two cases. Let $\sigma\in B^\Sigma_\rho(\bar\sigma)$ with $\rho<\hat\tau$. Then $\bar\sigma^i_{\rm on}+\hat\tau<\sigma^i_{\rm on}+2\hat\tau$ and by the choice of $\hat\tau$ we obtain for $t\in\,]\sigma^i_{\rm on},\bar\sigma^i_{\rm on}+\hat\tau[$ for (3.4) the solution

$$\zeta(t;\sigma^i_{\rm on},0,w) = f'(w)(t-\sigma^i_{\rm on}),\quad v(t)=w,$$

since g vanishes for $|x|<\varepsilon_g$. Hence, $\frac{\rm d}{{\rm d}w}\zeta(t;\sigma^i_{\rm on},0,w) = f''(w)(t-\sigma^i_{\rm on})\ge\beta(t-\sigma^i_{\rm on})$ by (5.10). For the remaining $t\in[\bar\sigma^i_{\rm on}+\hat\tau,\bar t+\tau]$ (i) follows from (5.10) by continuity for $\rho>0$ and $\tau>0$ small enough.

For (ii)–(v) we note that as a consequence for every $\sigma\in B^\Sigma_\rho(\bar\sigma)$ and $t\in\,]\sigma^i_{\rm on},\bar t+\tau]$ the mapping

$$w\in J_w\longmapsto\zeta(t;\sigma^i_{\rm on},0,w) \tag{5.11}$$

is strictly montone increasing and thus one-to-one. By the choice of $\hat\tau$ for $t\in\,]\sigma^i_{\rm on},\bar\sigma^i_{\rm on}+\hat\tau]$ we have as in (i) simply $W(t,x,\sigma) = f'^{-1}(\frac{x}{t-\sigma^i_{\rm on}})$. For $t\in[\bar\sigma^i_{\rm on}+\hat\tau,\bar t+\tau]$ by (i) the interval

$$\Big]\zeta(t;\sigma^i_{\rm on},0,w_l)-\frac{\beta\kappa}{2}(\hat\tau-\rho),\zeta(t;\sigma^i_{\rm on},0,w_r)+\frac{\beta\kappa}{2}(\hat\tau-\rho)\Big[$$

is contained in the image of (5.11), since $\hat{\tau}-\rho \leq \bar{\sigma}^i_{\text{on}}+\hat{\tau}-\sigma^i_{\text{on}}$ and hence also $[\xi_l(t),\xi_r(t)]$ is in the image of (5.11) if $\rho>0$ is chosen sufficiently small. The Lipschitz continuity of $W(\cdot,\sigma)$ follows directly from (i) and the Lipschitz continuity of $Y_R(\cdot,\sigma)$ is a consequence of (3.4). The differentiability of the considered mappings and the derivative formulas in (iv), (v) follow from the implicit function theorem. ■

LEMMA 5.7 *Let* (A3), (A4) *hold, consider* u_I *as in* (3.1), $\bar{\sigma}\in\Sigma$ *and let* $\bar{x}$ *be a continuity point of* $y(\bar{t},\cdot;\bar{\sigma})$ *of class* R^c_S *with* $\bar{\xi}$ *approaching the rarefaction center* $(\bar{\sigma}^i_{\text{on}},0)$, *i.e.* $\bar{\xi}(\bar{\sigma}^i_{\text{on}})=0$. *Then the following holds.*

(i) There exists a maximal open interval $I\ni\bar{x}$, *such that* $\{\bar{t}\}\times I$ *contains no point of the shock set and that all backward characteristics through a point* $(\bar{t},x)\in\{\bar{t}\}\times I$ *end in* $(\bar{\sigma}^i_{\text{on}},0)$. *In particular, none of those characteristics intersects* $\{x=0\}$ *during another off-phase* $[\bar{\sigma}^s_{\text{off}},\bar{\sigma}^s_{\text{on}}]$ *with* $s\geq i+1$.

(ii) $y(\bar{t},\cdot;\bar{\sigma})$ *is continuously differentiable on* I.

(iii) Let $\hat{I}:=]x_l,x_r[$ *be an interval with* $x_l,x_r\in I$. *Denote by* $\xi_{l/r}$ *the genuine backward characteristics through* $(\bar{t},x_{l/r})$ *and set* $w_{l/r}:=v(\bar{\sigma}^i_{\text{on}};\bar{t},x_{l/r},y(\bar{t},x_{l/r};\bar{\sigma}))$. *Then there exist* $\kappa,\beta>0$, *such that* (5.10) *is satisfied.*

(iv) After the possible reduction of τ *from Lemma 5.6 there exists* $\rho>0$ *such that for every* $\sigma\in B^\Sigma_\rho(\bar{\sigma})$ *and* $s\in]\sigma^i_{\text{on}},\bar{t}[$ *with* $S=S(\tau,\sigma)$ *from Lemma 5.6 holds*

$$y(t,x;\sigma)=Y_R(t,x,\sigma)\quad\forall(t,x)\in S\cap\{t\geq s\}.\tag{5.12}$$

Proof. To show (i) assume that there is a sequence $x_k\to\bar{x}$ of continuity points, such that the genuine bachward characteristic ξ_k through $(\bar{t},x_k)$ does not end in $(\bar{\sigma}^i_{\text{on}},0)$. Since ξ_k and $\bar{\xi}$ may not intersect at some time $t>\bar{\sigma}^i_{\text{on}}$, ξ_k reaches $t=\bar{\sigma}^i_{\text{on}}$ and by the backward stability of solutions (ζ,v) to (3.4) we have $\xi_k(\bar{\sigma}^i_{\text{on}})\to\xi(\bar{\sigma}^i_{\text{on}})$ and $y(\bar{\sigma}^i_{\text{on}},\xi_k(\bar{\sigma}^i_{\text{on}});\bar{\sigma})= v(\bar{\sigma}^i_{\text{on}};\bar{t},x_k,y(\bar{t},x_k;\bar{\sigma}))\to v(\bar{\sigma}^i_{\text{on}};\bar{t},\bar{x},y(\bar{t},\bar{x};\bar{\sigma}))=y(\bar{\sigma}^i_{\text{on}}+,0;\bar{\sigma})=\bar{w}\notin\{0,1\}$. But for k large enough we have $y(\bar{\sigma}^i_{\text{on}},\xi_k(\bar{\sigma}^i_{\text{on}});\bar{\sigma})\in\{0,1\}$ by (A4) and Lemma 3.5, which yields a contradiction. Hence, (i) holds in an open neighborhood of $\bar{x}$ and thus there exists a maximal interval I as asserted in (i) and every point $(\bar{t},x)$, $x\in I$ is of class R^c.

By (i) and its proof every point $(\bar{t},x)$, $x\in I$ is of class R^c and thus $y(\bar{t},x;\bar{\sigma})= Y_R(\bar{t},x,\bar{\sigma})$ with Y_R from Lemma 5.6 and thus (ii) follows from Lemma 5.6, (iii).

Let $\hat{I}$ be as in (iii). By continuity for every $x\in[x_l,x_r]$ we find an open neighborhood and $\tilde{\kappa},\tilde{\beta}>0$ such that (5.10) is satisfied. We choose a finite covering to obtain $\kappa,\beta>0$ such that (5.10) holds for the whole interval $\hat{I}$.

By (iii) we can apply Lemma 5.6 and obtain $S=S(\tau,\sigma)$ and $\rho>0$ such that there exists a function Y_R with $Y_R(\cdot,\sigma)\in C^1(S\cap\{t\geq s\})$ for all $\sigma\in B^\Sigma_\rho(\bar{\sigma})$ and all $s>\sigma^i_{\text{on}}$ that describes a rarefaction wave centered at $(\sigma^i_{\text{on}},0)$. If $\eta>0$ is chosen sufficiently small, we have $[x_l-2\eta,x_r+2\eta]\subset I$ and we can reapply Lemma 5.6 to $]x_l-\eta,x_r+\eta[$ instead of $\hat{I}$ obtaining smaller values $\hat{\tau},\tau,\rho$, a larger interval $\tilde{J}_w\supset J_w$ and a new stripe $\tilde{S}\supset S$. Since (A4) holds, we can after a possible reduction of $\rho>0$ also apply Lemma 3.5 (v) and obtain $y(t,x;\sigma)=f'^{-1}(\frac{x}{t-\sigma^i_{\text{on}}})=Y_R(t,x,\sigma)$ for all $\sigma\in B^\Sigma_\rho(\bar{\sigma})$ and all $t\in]\sigma^i_{\text{on}},\bar{\sigma}^i_{\text{on}}+\tilde{\tau}[$ and $x\in]f'(0)(t-\sigma^i_{\text{on}}),f'(1)(t-\sigma^i_{\text{on}})[$.

By construction (5.12) holds for $\bar{\sigma}$ and as just shown also for $\sigma\in B^\Sigma_\rho(\bar{\sigma})$ and $t\in]\sigma^i_{\text{on}},\bar{\sigma}^i_{\text{on}}+\tilde{\tau}[$. To establish (5.12) it is enough to show that possibly after reducing $\rho>0$ for all $\sigma\in B^\Sigma_\rho(\bar{\sigma})$ there are continuity points $\tilde{x}_l\in]x_l-2\eta,x_l-\eta[$ and $\tilde{x}_r\in]x_r+\eta,x_r+2\eta[$ such that the backward characteristics $\zeta(t;\bar{t},\tilde{x}_{l/r},y(\bar{t},\tilde{x}_{l/r};\sigma))$ meet $t=\bar{\sigma}^i_{\text{on}}+\tilde{\tau}$ in $I_\sigma:=]f'(0)(\bar{\sigma}^i_{\text{on}}+\tilde{\tau}-\sigma^i_{\text{on}}),f'(1)(\bar{\sigma}^i_{\text{on}}+\tilde{\tau}-\sigma^i_{\text{on}})[$. In fact, since genuine backward characteristics may intersect only at their end points, this ensures that all backward characteristics through continuity points $x \in]x_l - \eta, x_r + \eta[$ meet $t = \bar{\sigma}^i_{\text{on}} + \tilde{\tau}$ in I_σ and thus (5.12) holds on $\tilde{S} \cap \{s \le t \le \bar{t}\}$ and thus for $\tau < \eta / \max\{f'(0), f'(1)\}$ also on $\tilde{S} \cap \{s \le t\}$.

Assume the contrary. Then there exists a sequence $(\sigma_k) \subset B^\Sigma_\rho(\bar{\sigma})$, $\sigma_k \to \bar{\sigma}$ such that $\zeta(\bar{\sigma}^i_{\text{on}} + \tilde{\tau}; \bar{t}, x, y(\bar{t}, x; \sigma_k)) \notin I_{\sigma_k}$ for all continuity points x of $y(\bar{t}, \cdot; \sigma_k)$ in $]x_l - 2\eta, x_l - \eta[$ or in $]x_r + \eta, x_r + 2\eta[$. By the $\mathrm{L}^1_{\text{loc}}$-stability from Corollary 3.1 we can select a subsequence such that $y(\bar{t}, \cdot; \sigma_k) \to y(\bar{t}, \cdot; \bar{\sigma})$ a.e. on $]x_l - 2\eta, x_r + 2\eta[$. Since the union of all points of discontinuity of $y(\bar{t}, \cdot; \sigma_k)$ has measure zero, we thus find continuity points $\tilde{x}_l \in]x_l - 2\eta, x_l - \eta[$, $\tilde{x}_r \in]x_l - 2\eta, x_l - \eta[$ of all $y(\bar{t}, \cdot; \sigma_k)$ with $y(\bar{t}, \tilde{x}_{l/r}; \sigma_k) \to y(\bar{t}, \tilde{x}_{l/r}; \bar{\sigma})$ and hence $\zeta(\bar{\sigma}^i_{\text{on}} + \tilde{\tau}; \bar{t}, \tilde{x}_{l/r}, y(\bar{t}, \tilde{x}_{l/r}; \sigma_k)) \to \zeta(\bar{\sigma}^i_{\text{on}} + \tilde{\tau}; \bar{t}, \tilde{x}_{l/r}, y(\bar{t}, \tilde{x}_{l/r}; \bar{\sigma})) = f'(\tilde{w}_{l/r})\tilde{\tau}$ for some $\tilde{w}_{l/r} \in]0, 1[$. Since there exist open neighborhoods $J_{l/r}$ of $f'(\tilde{w}_{l/r})\tilde{\tau}$ with $J_{l/r} \subset I_{\sigma_k}$ we see that for k large enough the backward characteristics satisfy $\zeta(\bar{\sigma}^i_{\text{on}} + \tilde{\tau}; \bar{t}, \tilde{x}_{l/r}, y(\bar{t}, \tilde{x}_{l/r}; \sigma_k)) \in I_{\sigma_k}$, which yields a contradiction. ■

We turn now to continuity points of class RB^c_S.

Lemma 5.8 *Let* (A3), (A4) *hold, consider* u_I *as in* (3.1), $\bar{\sigma} \in \Sigma$ *and let* $\bar{x}$ *be a* RB^c_S*-point of* $y(\bar{t}, \cdot; \bar{\sigma})$ *on the left boundary of a rarefaction wave, i.e.* $\bar{\xi}(\bar{\sigma}^i_{\text{on}}) = 0$, $\bar{w} = 0$. *Then the following holds.*

(i) There exists a maximal open interval $I \ni \bar{x}$, *such that* $\{\bar{t}\} \times I$ *contains no point of the shock set and that all backward characteristics through a point* $(\bar{t}, x) \in \{\bar{t}\} \times I$ *end in* $]\bar{\sigma}^i_{\text{off}}, \bar{\sigma}^i_{\text{on}}] \times \{0\}$. *In particular, none of those characteristics intersects* $\{x = 0\}$ *during another off-phase* $[\bar{\sigma}^s_{\text{off}}, \bar{\sigma}^s_{\text{on}}]$ *with* $s \ge i + 1$.

(ii) $y(\bar{t}, \cdot; \bar{\sigma})$ *is continuously differentiable on* $I \setminus \bar{x}$ *and Lipschitz continuous on* I.

(iii) Let $\hat{I} :=]x_l, x_r[\ni \bar{x}$ *be an interval with* $x_l, x_r \in I$. *Denote by* $\xi_{l/r}$ *the genuine backward characteristics through* $(\bar{t}, x_{l/r})$ *and set* $\theta_l := \max\{t \in]\bar{\sigma}^i_{\text{off}}, \bar{\sigma}^i_{\text{on}}[\ : \zeta(t; \bar{t}, x_l, y(\bar{t}, x_l; \bar{\sigma})) = 0\}$, $w_r := v(\bar{\sigma}^i_{\text{on}}; \bar{t}, x_r, y(\bar{t}, x_r; \bar{\sigma}))$. *Then there exist* $\kappa, \beta > 0$, $w_l < \bar{w}$ *and* $\theta_r > \bar{\sigma}^i_{\text{on}}$ *such that* (5.9) *and* (5.10) *are satisfied.*

(iv) After a possible reduction of τ *from Lemmas 5.4 and 5.6 there exists* $\rho > 0$ *such that for every* $\sigma \in B^\Sigma_\rho(\bar{\sigma})$ *and* $s \in]\sigma^i_{\text{on}}, \bar{t}[$ *there holds*

$$y(t, x; \sigma) = \begin{cases} Y_C(t, x) & \text{if } x \le \zeta(t; \sigma^i_{\text{on}}, 0, 0), \\ Y_R(t, x, \sigma) & \text{else}, \end{cases} \qquad \forall (t, x) \in S \cap \{t \ge s\}, \quad (5.13)$$

where $S := \big(S_l \cap \{x \le \bar{\xi}\}\big) \cup \big(S_r \cap \{x \le \bar{\xi}\}\big)$ *with* S_l *and* S_r *obtained from Lemma 5.4 and 5.6, respectively.*

(v) The mapping

$$\sigma \in B^\Sigma_\rho(\bar{\sigma}) \longmapsto y(\bar{t}, \cdot; \sigma) \in \mathrm{L}^r(\hat{I}) \quad (5.14)$$

is continuously Fréchet-differentiable for all $r \in [1, \infty[$ *with derivative*

$$\mathrm{d}_\sigma y(\bar{t}, \cdot; \sigma) = \mathbb{1}_{x \ge \zeta(t; \sigma^i_{\text{on}}, 0, 0)} \, \mathrm{d}_\sigma Y_R(\bar{t}, \cdot, \sigma).$$

(vi) The mapping

$$\sigma \in B^\Sigma_\rho(\bar{\sigma}) \longmapsto y(\cdot; \sigma) \in \mathrm{C}(S \cap \{t \ge s\}) \quad (5.15)$$

is Lipschitz continuous.

Analogous results hold for continuity points $\bar{x}$ of class RB_S^c on the right boundary of the rarefaction wave, i.e. $\bar{\xi}(\bar{\sigma}_{\text{on}}^i) = 0$, $\bar{w} = 1$.

Proof. (i)-(iii) can be proven in a similar fashion as in Lemma 5.7.

For claim (iv) we remark that for $\varepsilon > 0$ sufficiently small, the intervals $\hat{I}_l :=]x_l, \bar{x} - \varepsilon[$ and $\hat{I}_r :=]\bar{x} + \varepsilon, x_r[$ contain only C_S^c and R_S^c points, respectively. Hence, Lemmas 5.5 and 5.7 are applicable and (5.13) holds on some smaller stripes $\tilde{S}_{l/r}$. Thus, the genuine backward characteristics through the remaining points in S cannot escape, which proves (5.13).

The differentiability of (5.14) and Lipschitz-continuity of (5.15) stated in (v) and (vi) follow from the differentiability and Lipschitz-continuity of Y_R, see Lemma 5.6, the regularity of Y_C, see Lemma 5.4, and the Lipschitz continuity of $\sigma \mapsto \zeta(t; \sigma_{\text{on}}^i, 0, 0)$, cf. Lemma 3.4. ∎

We remark that by assumption (A4) and the previously stated results in this section, for every off-switching point $\bar{\sigma}_{\text{off}}^i$ and genuine backward characteristic $\check{\xi}_i$ through $(\bar{\sigma}_{\text{off}}^i, 0)$ there exists a stripe $\check{S}_i$ around $\check{\xi}_i$, $\rho > 0$ and $s \in]0, \bar{\sigma}_{\text{on}}^i[$ such that the mappings

$$\sigma \in B_\rho^\Sigma(\bar{\sigma}) \longmapsto y(\cdot; (0, \sigma_{\text{off}}^1, \dots, \sigma_{\text{on}}^{i-1}, T)) \in C(\check{S}_i \cap \{t \geq s\}), \tag{5.16a}$$

$$(t, x) \in \check{S}_i \longmapsto y(t, x; (0, \bar{\sigma}_{\text{off}}^1, \dots, \bar{\sigma}_{\text{on}}^{i-1}, T)) \tag{5.16b}$$

are Lipschitz continuous. This fact will be of special interest in the consideration of CB_S^c-points in the following lemma.

LEMMA 5.9 *Let* (A3), (A4) *hold, consider u_I as in* (3.1), $\bar{\sigma} \in \Sigma$ *and let $\bar{x}$ be a CB_S^c-point with $\bar{\xi}$ approaching $(\bar{\sigma}_{\text{off}}^i, 0)$ from the left, i.e. $\bar{\xi}(\bar{\sigma}_{\text{off}}^i) = 0$, $\bar{w} = 0$.*

Then the following holds.

(i) There exists a maximal open interval $I \ni \bar{x}$, such that all points in $I \cap \{x < \bar{x}\}$ are of the same class $X_l \in \{X_I^c, R_S^c, C_S^s\}$ and all points in $I \cap \{x > \bar{x}\}$ are C_S^c-points.

(ii) $y(\bar{t}, \cdot; \bar{\sigma})$ is continuously differentiable on $I \setminus \{\bar{x}\}$ and Lipschitz continuous on I.

(iii) Let $\hat{I} :=]x_l, x_r[\ni \bar{x}$ be an interval with $x_l, x_r \in I$. Denote by $\xi_{l/r}$ the genuine backward characteristics through $(\bar{t}, x_{l/r})$ and set $\theta_r := \max\{t \in]\bar{\sigma}_{\text{off}}^i, \bar{\sigma}_{\text{on}}^i[\;:\; \zeta(t; \bar{t}, x_r, y(\bar{t}, x_r; \bar{\sigma})) = 0\}$. Then there exist $\kappa, \beta > 0$ and $\theta_l < \bar{\sigma}_{\text{off}}^i$ such that (5.9) *is satisfied. Thus we can apply Lemma 5.4. Denote by S_r the stripe and by Y_C the local solution therein.*

(iv) For each $x \in]x_l, \bar{x}[$ we can apply either Corollary 5.3, Lemma 5.4 or Lemma 5.6 and obtain local solutions Y^x on stripes S^x around the unique backward characteristics.

There exists an extension Y_l of these solutions Y^x, that is defined on a stripe S_l around $\bar{\xi}$, such that Y_l obeys the same continuity and stability properties than Y^x.

(v) Consider $S_{l/r}$ as above and set $S := \big(S_l \cap \{x \leq \bar{\xi}\}\big) \cup \big(S_r \cap \{x \geq \bar{\xi}\}\big)$ and define

$$\tilde{Y}(t, x, \sigma) := \begin{cases} Y_l(t, x; \sigma) & \text{if } x \leq \bar{\xi}(t), \\ Y_C(t, x) & \text{else}, \end{cases} \qquad \forall (t, x) \in S. \tag{5.17}$$

Then it holds for every $r \in [1, \infty[$ that

$$\lim_{\sigma \to \bar{\sigma}} \frac{\left\| \tilde{Y}(\bar{t}, \cdot, \sigma) - y(\bar{t}, \cdot; \sigma) \right\|_{r, \hat{I}}}{\|\sigma - \bar{\sigma}\|} = 0.$$

(vi) The mapping $\sigma \in \Sigma_{\mathrm{ad}} \mapsto y(\bar t, \cdot; \sigma) \in \mathrm{L}^r(\hat I)$ is Fréchet-differentiable in $\bar\sigma$.

Proof. Assertions (i)-(iii) can be proven in a similar fashion as in Lemma 5.6.

The precise proof of statement (iv) depends on the class X_l, but follows the same ideas for each case. We give the proof for $X_l = R_S^{\mathrm{c}}$ exemplarily. By assumption (5.8) holds for some $\bar\sigma_{\mathrm{on}}^j$, $j < i$, and every backward characteristic through $\{\bar t\} \times [x_l, \bar x]$. We set $w_l := v(\bar\sigma_{\mathrm{on}}^j; \bar t, x_l, y(\bar t, x_l; \bar\sigma))$. By continuity we find $\kappa, \beta > 0$ and $w_r > \bar w$ such that (5.10) is satisfied. Now, we can apply Lemma 5.6 to obtain S_l and Y_l.

For claim (v) we remark that by Corollary 5.3, Lemma 5.5 and Lemma 5.7 (once more depending on the class X_l) for $\varepsilon > 0$ sufficiently small we can construct smaller stripes $\tilde S_{l/r}$ based on the intervals $\hat I_l :=]x_l, \bar x - \varepsilon[$ and $\hat I_r :=]\bar x + \varepsilon, x_r[$ such that for σ sufficiently close to $\bar\sigma$ we have $\tilde Y(\bar t, \cdot, \sigma) = y(\bar t, \cdot; \sigma)$ on $\hat I_l \cup \hat I_r$. Moreover, we may let ε tend to zero as $\|\bar\sigma - \sigma\|$ does. Using again the non-intersection property of genuine characteristics we know that for σ sufficiently close to $\bar\sigma$, the backward characteristics through continuity points $x \in I_\varepsilon := [\bar x - \varepsilon, \bar x + \varepsilon]$ of $y(\bar t, \cdot; \sigma)$ may not escape. For such a point one simply verifies that for $z := \zeta(\bar\sigma_{\mathrm{off}}^i; \bar t, x, y(\bar t, x; \sigma))$ we have

$$\begin{aligned}
|y(\bar t, x; \sigma) - y(\bar t, x; \bar\sigma)| &\le \left| v(\bar t; \bar\sigma_{\mathrm{off}}^i, z, u_r(\sigma, z)) - v(\bar t; \bar\sigma_{\mathrm{off}}^i, z, u_r(\bar\sigma, z)) \right| \\
&\quad + L_{y(\bar t, \cdot; \bar\sigma)} \left| \zeta(\bar t; \bar\sigma_{\mathrm{off}}^i, z, u_r(\sigma, z)) - \zeta(\bar t; \bar\sigma_{\mathrm{off}}^i, z, u_r(\bar\sigma, z)) \right| \\
&\le \left(1 + L_{y(\bar t, \cdot; \bar\sigma)}\right) L_{\zeta, v} \left| u_r(\sigma, z) - u_r(\bar\sigma, z) \right|
\end{aligned}$$

with $\quad u_r(\sigma, z) := \begin{cases} y(\bar\sigma_{\mathrm{off}}^i, z; (0, \sigma_{\mathrm{off}}^1, \ldots, \sigma_{\mathrm{on}}^{i-1}, T)) & \text{if } z < (\bar\sigma_{\mathrm{off}}^i - \sigma_{\mathrm{off}}^i) f'(0), \\ 0 & \text{else.} \end{cases}$

By construction and (5.16) we obtain

$$|u_r(\sigma, z) - u_r(\bar\sigma, z)| \le \left(L_{y,\sigma} + |f'(0)| \cdot L_{y,x} \right) \|\bar\sigma - \sigma\|,$$

where $L_{y,\sigma}$ and $L_{y,x}$ denote the Lipschitz constants of (5.16a) and (5.16b), respectively.

Recall that by construction the function $\sigma \in B_\rho^\Sigma(\bar\sigma) \mapsto \tilde Y(\bar t, \cdot, \sigma) \in \mathrm{L}^r(\hat I)$ is Lipschitz continuous for some $\rho > 0$ with Lipschitz constant $L_{\tilde Y}$. We combine these results and obtain

$$\begin{aligned}
\left\| \tilde Y(\bar t, \cdot, \sigma) - y(\bar t, \cdot; \sigma) \right\|_{r, \hat I} &\le \| y(\bar t, \cdot; \sigma) - y(\bar t, \cdot; \bar\sigma) \|_{r, I_\varepsilon} + \left\| \tilde Y(\bar t, \cdot, \sigma) - \tilde Y(\bar t, \cdot, \bar\sigma) \right\|_{r, I_\varepsilon} \\
&\le \left(\left(1 + L_{y(\bar t, \cdot; \bar\sigma)}\right) L_{\zeta, v} \left(L_{y,\sigma} + |f'(0)| \cdot L_{y,x} \right) + L_{\tilde Y} \right) (2\varepsilon)^{\frac{1}{r}} \|\sigma - \bar\sigma\|.
\end{aligned}$$

Letting $\varepsilon, \|\bar\sigma - \sigma\|$ tend to zero concludes the proof of (v).

(vi) is a consequence of the differentiability of $\sigma \mapsto Y_l(\bar t, \cdot, \sigma) \in \mathrm{L}^r(S_l \cap \{t = \bar t\})$ and hence of $\sigma \mapsto \tilde Y_l(\bar t, \cdot, \sigma) \in \mathrm{L}^r(\hat I)$ combined with (v). ∎

5.3 *Differentiability at shock points*

Before we state the differentiability of the shock position, we show that it depends at least locally Lipschitz continuous on the control and separates two smooth parts of the solution.

Lemma 5.10 (Stability of the shock position) *Let* (A3), (A4) *hold and let u_I be as in* (3.1) *and $\bar\sigma \in \Sigma_{\mathrm{ad}}$. Furthermore, let $\bar x$ be a $X_l X_r$-point with $X_{l/r} \in \{X_I^{\mathrm{c}}, C_S^{\mathrm{c}}, R_S^{\mathrm{c}}\}$. Denote* *by $Y_{l/r}$ the local solution constructed on the stripe $S_{l/r}$ by applying Corollary 5.3, Lemma 5.4 or Lemma 5.6 to $\xi_{l/r}$.*

Then, there exists a neighborhood $I :=]x_l, x_r[$ of $\bar{x}$ such that the following holds:

(i) $y(\cdot;\bar{\sigma})$ is locally given by

$$y(t,x;\sigma) = \begin{cases} Y_l(t,x,\bar{\sigma}) & \text{if } (t,x) \in S_l \cap \{x < \eta(t)\}, \\ Y_r(t,x,\bar{\sigma}) & \text{if } (t,x) \in S_r \cap \{x > \eta(t)\}. \end{cases}$$

(ii) There exists $\rho > 0$ and a Lipschitz continuous function

$$x_s : \sigma \in B_\rho^\Sigma(\bar{\sigma}) \quad \longmapsto \quad x_s(\sigma) \tag{5.18}$$

with $x_s(\bar{\sigma}) = \bar{x}$, such that for all $\sigma \in B_\rho^\Sigma(\bar{\sigma})$ holds

$$y(\bar{t},x;\sigma) = \begin{cases} Y_l(\bar{t},x,\sigma) & \text{if } x \in]x_l, x_s(\sigma)[, \\ Y_r(\bar{t},x,\sigma) & \text{if } x \in]x_s(\sigma), x_r[. \end{cases}$$

Proof. The first assertion can be proven by using the backward stability of genuine backward characteristics according to Lemma 3.4.

A reinspection of the proof of [48, Lem. 6.2] shows, see [41, Lem. 6.3.1 & 7.3.1], that the class (C^c) of the extreme characteristics is not explicitly used, but only the results on the local solutions $Y_\pm$ on the stripes $S_\pm$, which also hold true for the current setting by Corollary 5.3, Lemma 5.4 or Lemma 5.6. ■

In the following lemma we consider an $X_I^c X_I^c$-shock, that has only one off-phase $]\sigma_{\text{off}}^i, \sigma_{\text{on}}^i[$ in its shock funnel. Then the point σ_{off}^i must be of class $\check{X}^i = X_I^c$, too. Afterwards we discuss how the result can be extended to general $X_l X_r$-shocks with $X_{l/r} \in \{X_I^c, C_S^c, R_S^c\}$, $\check{X}^i \in \{X_I^c, C_S^c, R_S^c, RB_S^c\}$.

Lemma 5.11 (Differentiability of the shock position) *Let the assumptions of Lemma 5.10 hold. Consider a shock point $x_s(\bar{\sigma}) = \bar{x}$ of class $X_I^c X_I^c$ with a single off-phase $]\bar{\sigma}_{\text{off}}^i, \bar{\sigma}_{\text{on}}^i[$ in its shock funnel. Let $\bar{\sigma}_{\text{off}}^i$ be nondegenerated according to Definition 4.4. Then for $\rho > 0$ sufficiently small the mapping* (5.18) *is continuously differentiable.*

Proof. Denote by $\check{\xi}$ the genuine backward characteristic through $(\bar{\sigma}_{\text{off}}^i, 0)$ and by $\check{S}^i$ and $\check{Y}^i$ the stripe and the local solution from Corollary 5.3. Let ρ be small enough such that $\check{S}^i \supset [\bar{\sigma}_{\text{off}}^i - \rho, \bar{\sigma}_{\text{off}}^i + \rho] \times I(0, \check{\xi}(\bar{\sigma}_{\text{off}}^i - \rho))$. Consider $\tilde{\delta} > 0$ and $s^i := \bar{\sigma}_{\text{on}}^i + \tilde{\tau}$ with $\tilde{\tau}$ from Lemma 3.5(v). Furthermore, let $\sigma \in B_\rho^\Sigma(\bar{\sigma})$ and set $\delta\sigma := \sigma - \bar{\sigma}$. By $\bar{y} := y(\cdot;\bar{\sigma})$, $y := y(\cdot;\sigma)$ we denote the respective solutions of (2.5) and by $\Delta y := y - \bar{y}$ their difference. As in [48, §8] one of the key points of the proof is the fact that for $\varepsilon > 0$ sufficiently small and $\hat{x}_{l/r} := x_s(\bar{\sigma}) \mp \varepsilon \in]x_l, x_r[$ (from Lemma 5.10) the following equality holds:

$$\int_{\hat{x}_l}^{\hat{x}_r} \Delta y(\bar{t},x)\ \mathrm{d}x = (x_s(\sigma) - x_s(\bar{\sigma}))[y(\bar{t}, x_s(\bar{\sigma}))] + O(\|\delta\sigma\|^2). \tag{5.19}$$

The above equation is obtained as in [48] using the Lipschitz continuity of $Y_{l/r}$ from Corollary 5.3 w.r.t. x, the Lipschitz continuity of (5.18) and the fact that $Y_{l/r}$ do not depend on σ.

The rest of this proof will be concerned with the derivation of an adjoint-based formula for the left hand side of (5.19). We avoid the introduction of a detailed analysis for linear transport equations with discontinuous coefficients on sliced domains, such as $\breve{\Omega}_{\bar{t}}$. Instead we show how the considered equation can be modified so that the results of [48, §7] can be used. A detailed description of the utilized localizing arguments can be found in [41, §4.2]. We denote by $\hat{\zeta}_{l/r}$ the genuine backward characteristics through $(\bar{t}, \hat{x}_{l/r})$ and by

$$\tilde{D}_0 := \{(t,x) \in [s^i, \bar{t}] \times \mathbb{R} \;:\; \hat{\zeta}_l(t) \leq x \leq \hat{\zeta}_r(t)\}$$

the area confined by them, see Figure 1. For $(t,x) \in \tilde{D}_0$ we define

$$a(t,x) := f'(\bar{y}(t,x)), \qquad \tilde{a}(t,x) := \int_0^1 f'(\bar{y}(t,x) + \lambda \Delta y(t,x))\,\mathrm{d}\lambda,$$
$$b(t,x) = \tilde{b}(t,x) := g_y(t,x,\bar{y}(t,x)).$$

Using the above abbreviations and the assumption that g is affine linear w.r.t y, we deduce that on $\tilde{D}_0$ the difference of y and $\bar{y}$ is a weak solution of

$$\partial_t \Delta y + \partial_x(\tilde{a}\Delta y) = \tilde{b}\Delta y. \tag{5.20}$$

We extend the functions $a, \tilde{a}, b, \tilde{b}$ to $[s^i, \bar{t}] \times \mathbb{R}$ by setting

$$(\tilde{a}, \tilde{b})(t,x) = (a,b)(t,x) = \begin{cases} (M_{f'}, b(t, \hat{\zeta}_l(t)+)) & \text{if } x < \hat{\zeta}_l(t), \\ (-M_{f'}, b(t, \hat{\zeta}_r(t)-)) & \text{if } x > \hat{\zeta}_r(t), \end{cases} \tag{5.21}$$

with $M_{f'}$ from (3.6). On $\tilde{D}_0$ the adjoint state p according to Definition 4.3 can be interpreted as the restriction of the reversible solution to the adjoint equation (4.3) on $[s^i, \bar{t}] \times \mathbb{R}$ in the sense of [48, Def. 7.5] for the same end data.

We define $\tilde{p}$ to be the reversible solution of the averaged adjoint equation

$$\partial_t \tilde{p} + \tilde{a}\partial_x \tilde{p} = -\tilde{b}\tilde{p}, \quad \tilde{p}(\bar{t}, \cdot) = p^{\bar{t}} \equiv \frac{1}{[y(\bar{t}, x_s(\bar{\sigma}); \bar{\sigma})]} \tag{5.22}$$

on $[s^i, \bar{t}] \times \mathbb{R}$ in the sense of [48, Def. 7.5]. We multiply (5.20) by $\tilde{p}$ and apply integration by parts on $\tilde{D}_0$, which yields

$$\begin{aligned} &\int_{\hat{x}_l}^{\hat{x}_r} \tilde{p}(\bar{t},x)\Delta y(\bar{t},x)\,\mathrm{d}x = \iint_{\tilde{D}_0} \Delta y(\partial_t \tilde{p} + \tilde{a}\partial_x \tilde{p} + \tilde{b}\tilde{p})\,\mathrm{d}x\,\mathrm{d}t \\ &+ \int_{s^i}^{\bar{t}} \tilde{p}(t, \hat{\zeta}_l(t))(-f'(y)\Delta y + f(y) - f(\bar{y}))(t, \hat{\zeta}_l(t))\;\mathrm{d}t \\ &+ \int_{s^i}^{\bar{t}} \tilde{p}(t, \hat{\zeta}_r(t))(f'(y)\Delta y - f(y) + f(\bar{y}))(t, \hat{\zeta}_r(t))\,\mathrm{d}t + \int_{\hat{\zeta}_l(s^i)}^{\hat{\zeta}_r(s^i)} \tilde{p}(s^i,x)\Delta y(s^i,x)\,\mathrm{d}x \\ &= \int_{\hat{\zeta}_l(s^i)}^{\hat{\zeta}_r(s^i)} p(s^i,x)\Delta y(s^i,x)\,\mathrm{d}x + o(\|\delta\sigma\|) \end{aligned} \tag{5.23}$$

The first integral in the middle part of the above equation vanishes since by [48, Thm. 7.7] $\tilde{p}$ solves (5.22) almost everywhere on $\tilde{D}_0$, the left and right boundary integrals vanish, since y is independent of the control σ and therefore y and $\bar{y}$ coincide locally. Finally, [48, Thm. 7.8] yields $\tilde{p} \to p$ in $\mathrm{C}(\tilde{D}_0)$ and by Corollary 3.1 $\Delta y(s, \cdot) \to 0$ in $\mathrm{L}^1_{\mathrm{loc}}$, which shows the second equality. Let

$$[\tilde{x}_l, \tilde{x}_r] \subset] - \tilde{\delta} + M_{f'}(s^i - \bar{\sigma}^i_{\mathrm{on}}), \tilde{\delta} - M_{f'}(s^i - \bar{\sigma}^i_{\mathrm{on}})[\setminus [f'(0)(s^i - \bar{\sigma}^i_{\mathrm{on}}), f'(1)(s^i - \bar{\sigma}^i_{\mathrm{on}})].$$

We split the integral on the righthand side of (5.23) into three parts:

$$\begin{aligned}
&\int_{\hat{\zeta}_l(s^i)}^{\hat{\zeta}_r(s^i)} p(s^i, \cdot)\Delta y(s^i, \cdot)\,\mathrm{d}x \\
&\quad = \int_{\hat{\zeta}_l(s^i)}^{\tilde{x}_l} p(s^i, \cdot)\Delta y(s^i, \cdot)\,\mathrm{d}x + \int_{\tilde{x}_l}^{\tilde{x}_r} p(s^i, \cdot)\Delta y(s^i, \cdot)\,\mathrm{d}x + \int_{\tilde{x}_r}^{\hat{\zeta}_r(s^i)} p(s^i, \cdot)\Delta y(s^i, \cdot)\,\mathrm{d}x
\end{aligned}$$

The middle part of the above integral can be computed using Lemmas 5.7 and 5.8.

We now show how to compute the first part, with the third being similar. Consider $\hat{t} := s^i - f'(0)\tilde{x}_l$ and $\sigma_+ := \bar{\sigma}^i_{\mathrm{off}} + \max(\delta\sigma^i_{\mathrm{off}}, 0)$ and $\sigma_- := \bar{\sigma}^i_{\mathrm{off}} + \min(\delta\sigma^i_{\mathrm{off}}, 0)$. Let $\tilde{D} := \tilde{D}_1 \cup \tilde{D}_2 \cup \tilde{D}_3$ with

$$\begin{aligned}
\tilde{D}_1 &:= \{(t, x) \in [\hat{t}, s^i] \times \mathbb{R}^- \,:\, \hat{\zeta}_l(t) \le x \le f'(0)(t - \hat{t})\}, \\
\tilde{D}_2 &:= \{(t, x) \in [\sigma_-, \hat{t}] \times \mathbb{R}^- \,:\, \hat{\zeta}_l(t) \le x\}, \\
\tilde{D}_3 &:= \{(t, x) \in [0, \sigma_-] \times \mathbb{R} \,:\, \hat{\zeta}_l(t) \le x \le \check{\xi}(t)\}.
\end{aligned}$$

We consider the reversible solutions $\tilde{p}, p$ of

$$\begin{aligned}
&\partial_t p + a\partial_x p = -bp, \qquad && p(s^i, \cdot) = p(s^i+, \cdot) \\
&\partial_t \tilde{p} + \tilde{a}\partial_x \tilde{p} = -\tilde{b}\tilde{p}, \qquad && \tilde{p}(s^i, \cdot) = p(s^i+, \cdot)
\end{aligned}$$

on $\tilde{D}$, where we extend the coefficients outside $\tilde{D}$ analogous to (5.21). Integration by parts on $\tilde{D}$ yields

$$\int_{\hat{\zeta}_l(s^i)}^{\tilde{x}_l} p(s^i, x)\Delta y(s^i, x)\,\mathrm{d}x = I_1 + I_2 + I_3 + I_4 + I_5 + I_6 + I_7 + I_8, \tag{5.24}$$

where

$$\begin{aligned}
I_1 &:= \iint_{\tilde{D}} \Delta y(\partial_t \tilde{p} + \tilde{a}\partial_x \tilde{p} + \tilde{b}\tilde{p})\,\mathrm{d}x\,\mathrm{d}t, \\
I_2 &:= \int_0^{s^i} \tilde{p}(t, \hat{\zeta}_l(t))(-f'(\bar{y})\Delta y + f(y) - f(\bar{y}))(t, \hat{\zeta}_l(t))\,\mathrm{d}t, \\
I_3 &:= \int_{\hat{\zeta}_l(0)}^{\check{\xi}(0)} \tilde{p}(0+, x)\Delta y(0+, x)\,\mathrm{d}x, \\
I_4 &:= \int_0^{\sigma_-} \tilde{p}(t, \check{\xi}(t))(f'(\bar{y})\Delta y - f(y) + f(\bar{y}))(t, \check{\xi}(t))\,\mathrm{d}t, \\
I_5 &:= \int_{\check{\xi}(\sigma_-)}^{0} \tilde{p}(\sigma_-, \cdot)\Delta y(\sigma_-, \cdot)\,\mathrm{d}x,
\end{aligned}$$

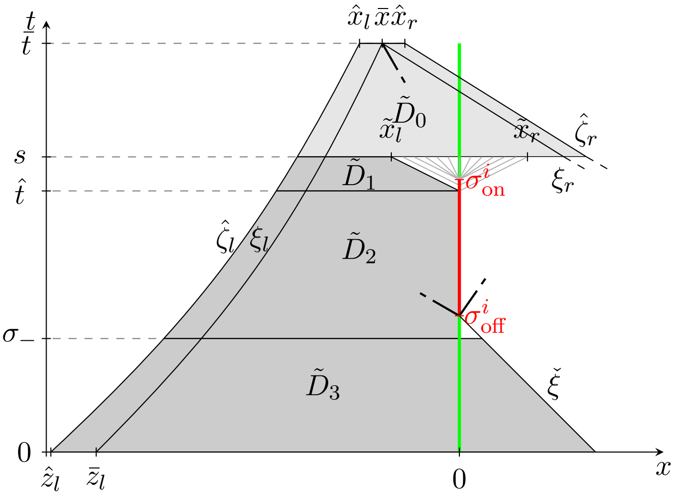

Figure 1. Illustration of the proof of Lemma 5.11

$$
\begin{aligned}
I_6 &:= \int_{\sigma_-}^{\sigma_+} \tilde{p}(t, 0-)(f(\bar{y}(t, 0-)) - f(y(t, 0-)))\,\mathrm{d}t,\\
I_7 &:= \int_{\sigma_+}^{\hat{t}} \tilde{p}(t, 0-) \cdot 0\,\mathrm{d}t,\\
I_8 &:= \int_{\hat{t}}^{s^i} \tilde{p}(t, f'(0)\cdot(t-\hat{t}))(f'(\bar{y})\Delta y - f(y) + f(\bar{y}))(t, f'(0)\cdot(t-\hat{t}))\,\mathrm{d}t.
\end{aligned}
$$

The domain of integration is also illustrated in Figure 1 with the boundary integrals I_2 to I_8 labeled counterclockwise. Since $\tilde{p}$ is an almost everywhere solution of the averaged adjoint equation, I_1 vanishes. Furthermore, $I_2 = I_4 = I_5 = 0$ because the local solutions Y around $\hat{\zeta}_l$ and $\check{\xi}$ are independent of $\delta\sigma$. I_3 vanishes, too, because the initial data are independent of the control. Obviously, $I_7 = 0$ and by Lemma 3.5 also $I_8 = 0$. Finally we have

$$
I_6 = p(\bar{\sigma}^i_{\text{off}}, 0-) \int_{\sigma_-}^{\sigma_+} (f(\bar{y}(t, 0-)) - f(y(t, 0-)))\,\mathrm{d}t + o(\|\delta\sigma\|)
$$

by the convergence of the averaged adjoint state. And depending on the sign of $\delta\sigma^i_{\text{off}}$ we have for $t \in [\sigma_-, \sigma_+]$

$$
f(\bar{y}(t, 0-)) - f(y(t, 0-)) = \begin{cases} 0 - f(\check{Y}^i(t, 0)), & \text{if } \delta\sigma^i_{\text{off}} > 0,\\ f(\check{Y}^i(t, 0)) - 0, & \text{if } \delta\sigma^i_{\text{off}} < 0, \end{cases}
$$

where we used, that $\check{Y}^i$ is independent of the control σ. The continuity of $\check{Y}^i$ on $\check{S}^i$ now shows

$$
\begin{aligned}
I_6 &= p(\bar{\sigma}^i_{\text{off}}, 0) \int_{\sigma_-}^{\sigma_+} \operatorname{sgn}(\delta\sigma^i_{\text{off}})(-f(\check{Y}^i(t, 0)))\,\mathrm{d}t + o(\|\delta\sigma\|)\\
&= p(\bar{\sigma}^i_{\text{off}}, 0) \int_{\sigma_-}^{\sigma_+} \operatorname{sgn}(\delta\sigma^i_{\text{off}})(-f(\check{Y}^i(\bar{\sigma}^i_{\text{off}}, 0)))\,\mathrm{d}t + o(\|\delta\sigma\|)
\end{aligned}
$$

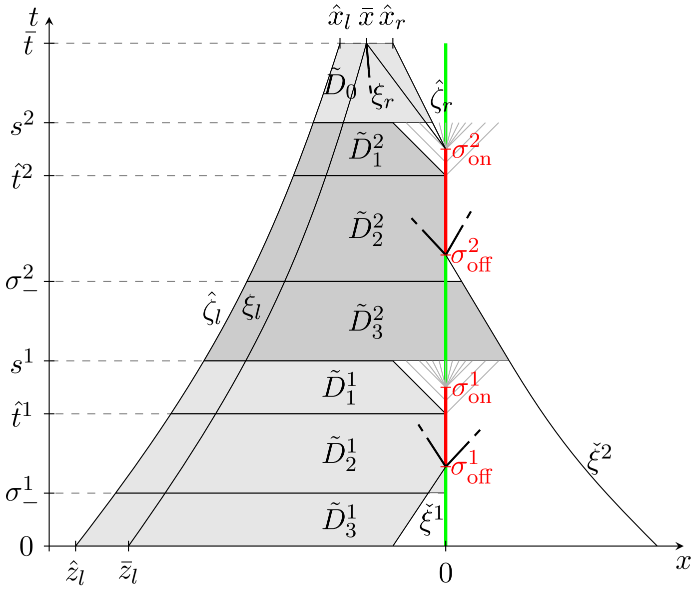

Figure 2. Illustration of Remark 5.12

$$= p(\bar{\sigma}_{\rm off}^i, 0)(-f(\check{Y}^i(\bar{\sigma}_{\rm off}^i, 0))) \cdot \delta\sigma_{\rm off}^i + o(\|\delta\sigma\|).$$

The continuity of the derivative $\frac{\rm d}{{\rm d}\sigma}x_s(\sigma)$ is a consequence of the stability of the solution p of the adjoint equation w.r.t. small perturbations in the coefficients and the fact that p is continuous in a neighborhood of $\bar{\sigma}_{\rm off}^i$, see [48, Thm. 7.7]. ■

Remark 5.12 The results of Lemma 5.11 are also valid if one considers the following modifications:

(i) If there are further off-phases in the shock funnel, the integration on $\tilde{D}$ is only applied up to time s^{i-1}. This means, that the integral I_3 in (5.24) does no longer vanish, but must be computed similar as the righthand side of (5.23) by reapplication of the above explained procedure, see Figure 2.

(ii) If one or more of the classes $\check{X}^i, X_{l/r}$ is C_S^c, the proof of Lemma 5.11 can easily be adapted. One can use exactly the same arguments. The only difference is the shape of the areas $\tilde{D}_i$ confined by the respective characteristics.

(iii) If one or more of the classes $\check{X}^i, X_{l/r}$ is R_S^c or $\check{X}^i = RB_S^c$, cf. ξ_r in Figure 2, the proof of Lemma 5.11 needs some further modifications: Since in (5.19) $Y_{l/r}$ may depend on the control σ, we have to use the Lipschitz continuity of the local solution $\sigma \mapsto Y_{l/r}(\cdot, \sigma)$ from Lemma 5.6, yielding

$$\int_{\hat{x}_l}^{\hat{x}_r} \Delta y(\bar{t}, x)\ {\rm d}x = (x_s(\sigma) - x_s(\bar{\sigma}))[y(\bar{t}, x_s(\bar{\sigma}))] + O(\|\delta\sigma\|(\varepsilon + \|\delta\sigma\|)). \qquad (5.25)$$

Moreover, all integrals along characteristics, i.e. the second and third integral in the middle part of (5.23), as well as I_2 and I_4 in (5.24), need no longer be equal to zero, but are $O(\|\delta\sigma\|^2)$. This can be shown by overestimating the remainder of the Tailor expansion by $O(\|Y(\cdot, \bar{\sigma}) - Y(\cdot, \sigma)\|^2_{{\rm C}(S)})$ and using the Lipschitz continuity of $\sigma \mapsto Y(\cdot, \sigma)$ with $Y = Y_R$ from Lemma 5.6 or Y as in (5.13) in Lemma 5.8.

5.4 *Proof of Theorem 4.6 and Theorem 4.8*

We finally have established all ingredients to prove our main theorems by piecing together the results of this section.

Proof of Theorem 4.6. In a first step we find $\rho > 0$ and neighborhoods I_j of the shock-points $\bar{x}_j$ such that on each of them Lemma 5.10 is applicable and (5.18) is continuously differentiable, see Lemma 5.11 and Remark 5.12. From this we obtain continuous shift differentiability of $\sigma \in B_\rho^\Sigma(\bar{\sigma}) \mapsto \mathrm{L}^1(\bigcup_{j=1}^{\bar{N}} I_j)$.

If there exist CB_S^c-points $x_{\hat{j}} \in [a,b]$, we find neighborhoods $\hat{I}_{\hat{j}}$ of $x_{\hat{j}}$, on which we apply Lemma 5.9 (vi).

It remains to consider the compact set $K := [a,b] \setminus \left(\bigcup_{j=1}^{\bar{N}} I_j \cup \bigcup_{\hat{j}} \hat{I}_{\hat{j}} \right)$ of continuity points. For each of these points x we can apply one of the Lemmas in §5.2 yielding an interval $\hat{I} = \hat{I}(x)$ such that the solution $y(\bar{t}, \cdot; \sigma) \in \mathrm{L}^r(\hat{I})$ depends continuously differentiably on the control $\sigma \in B_\rho^\Sigma(\bar{\sigma})$ for $\rho > 0$ sufficiently small. We can choose a finite covering $\bigcup_{x \in F} \hat{I}(x) \supset K$ and reduce $\rho > 0$ such that the assertion of Theorem 4.6 is proven to hold. ■

Proof of Theorem 4.8. The proof is mainly a combination of the one of Lemma 5.11 and [49, Thm. 5]. Basically, the adjoint calculus from Lemma 5.11 is used on the whole domain in order to find a first order approximation of $\int_\Omega \bar{\psi}_y \Delta y \,\mathrm{d}x$ instead of the lefthand side of (5.19) or (5.25), respectively. Working in the setting of a single off-phase, one obtains

$$\begin{aligned} \hat{J}'(\bar{\sigma}) \cdot \delta\sigma = \int_{f'(0)(s^i - \bar{\sigma}^i_{\mathrm{on}})}^{f'(1)(s^i - \bar{\sigma}^i_{\mathrm{on}})} p(s^i, x) \cdot (f'^{-1})' \left(\frac{x}{s^i - \bar{\sigma}^i_{\mathrm{on}}} \right) \frac{x}{(s^i - \bar{\sigma}^i_{\mathrm{on}})^2} \,\mathrm{d}x \cdot \delta\sigma^i_{\mathrm{on}} \\ + \big(p(\sigma_{\mathrm{off}}, 0+) - p(\bar{\sigma}^i_{\mathrm{off}}, 0-)\big) \cdot f(y(\bar{\sigma}^i_{\mathrm{off}}, 0) \cdot \delta\sigma^i_{\mathrm{off}}. \end{aligned}$$

We consider the integral in the above formula and make some change of variables:

$$\begin{aligned} \int_{f'(0)(s^i - \bar{\sigma}^i_{\mathrm{on}})}^{f'(1)(s^i - \bar{\sigma}^i_{\mathrm{on}})} p(s^i, x) \cdot (f'^{-1})' \left(\frac{x}{s^i - \bar{\sigma}^i_{\mathrm{on}}} \right) \frac{x}{(s^i - \bar{\sigma}^i_{\mathrm{on}})^2} \,\mathrm{d}x \\ = \int_{f'(0)}^{f'(1)} p(s^i, (s^i - \bar{\sigma}^i_{\mathrm{on}})w) \cdot w \cdot (f'^{-1})'(w) \,\mathrm{d}w. \end{aligned}$$

The time $s^i > \bar{\sigma}^i_{\mathrm{on}}$ is ensured to exist by Lemma 3.5, but is not an a priori known quantity. Since we are in the ε_g-neighborhood of $x = 0$, all characteristics are straight lines and p is constant along them. Therefore, we can pass to the limit $s \searrow \bar{\sigma}^i_{\mathrm{on}}$ and obtain the formula presented in Theorem 4.8. ■

6. Conclusion and future work

In this paper we have analyzed the differentiability of the reduced objective function for optimal control problems, where the state is governed by a hyperbolic conservation law on a simple switched network with on/off-switching. Based on an appropriate adjoint calculus we were able to deal with shocks in the entropy solution and to show that the state depends shift-differentiably on the switching times of the node condition. Here, we were able to allow arbitrary shock formations, the only restriction was the requirement (which was shown to hold for almost all times) that at the observation time $\bar{t}$ there are on $[a, b]$ no shock generation points and only a finite number of shocks, that all are neither degenerated nor shock interaction points. By applying [48, Lem. 2.3] we were able to deduce the differentiability of reduced objective function from that result. Based on the introduced adjoint calculus we have also derived a formula for the gradient of the reduced objective. The result of this paper forms the basis for the application of gradient-based optimization methods to such problems.

One can straight forward extend the presented analysis to the case where the initial data are additionally controlled. The same holds, if one considers several positions of on/off-switching devices in a row, or even nodes with multiple incoming and outgoing edges with a modal node condition, that either suspends the inflow from or the outflow into certain arcs or directly connects pairs of in- and outgoing arcs time dependently.

In the future we want to investigate how to deal with situations where the assumptions on the off-switching points are no longer satisfied. This means that we have to explore the case where no shock is introduced at the artificial boundary during the whole off-phase and the case where the location of the switching device, i. e. $x = 0$, is not a continuity point of $y(\sigma^i_{\text{off}}, \cdot)$. In the first case, the difficulty arises from the fact the small perturbations of the control may entirely change the structure in the neighborhood of the considered off-phase: There might be two shocks emanating from the artificial boundary or the off-switching point, and a rarefaction wave at the corresponding on-switching point. For the second case, we expect directional differentiability for the shock position and the reduced objective function to hold. This can be seen from I_6 in Lemma 5.11, for which the limit $\delta\sigma^i_{\text{off}} \to 0$ depends on the the direction in which the incoming shock at $(\bar{\sigma}^i_{\text{on}}, 0)$ moves.

Further questions are for example, whether one may choose different fluxes on the different edges and if one can combine the problem with more common node conditions allowing for splitting and merging of incoming and outgoing flow, as those from [12, 14].

It would be of practical interest to investigate whether the considered approach is extendable to systems of conservation laws where one has multiple conserved quantities, as in models for gas or water pipelines.

The presented sensitivity and adjoint calculus can be used for the numerical approximation of the optimal control problem under consideration. For the Cauchy problem there exist several works on the convergence of optimal solutions of discretized optimal control problems, e.g. [13, 46], and the convergence of sensitivities, adjoints and reduced gradients, see [22, 23, 47–49] and also [13] for an alternating descent method.

Our current investigations focus on the extension of those convergence results to the problem considered in this paper. The main question arising in this context is the appropriate discrete approximation of the shift of switching times between on- an off-phases. Here, we will consider and compare two different approaches. In the first one, we consider the variation of the times step sizes between switching times, while for the latter we want to use fixed time steps.

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