Optimal Boundary Control of Hyperbolic Balance Laws with State Constraints [*]

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Abstract. In this paper we analyze the optimal control of initial-boundary value problems for entropy solutions of scalar hyperbolic balance laws with pointwise state constraints. Hereby, we suppose that the initial and the boundary data switch between different C^1-functions at certain switching points, where the C^1-functions and the switching points are considered as the control. For a class of cost functionals, we prove first order necessary optimality conditions for the corresponding optimal control problem with state contraints. Furthermore, we use a Moreau-Yosida type regularization to approximate the optimal control problem with state constraints. We derive optimality conditions for the regularized problems and finally prove convergence to the solution of the optimal control problem with state constraints.

Key words. optimal control, scalar conservation law, state constraints, Moreau-Yosida

AMS subject classifications. 49K20, 35L65, 35R05, 49J50

1. Introduction. In this paper we derive necessary optimality conditions for state constrained optimal control problems of the form

$$\text{(P)} \qquad \min J(y(w)) := \int_a^b \psi(y(\bar{t}, x; w), y_d(x)) \, \mathrm{d}x + R(w),$$

where $\psi \in C^{1,1}_{\text{loc}}(\mathbb{R}^2)$, $y_d \in BV_{loc}(\mathbb{R}) \cap L^\infty(\mathbb{R})$ is the desired state, $R : W \to [0, \infty)$ is a Fréchet differentiable regularization term and y is given by the solution of the following initial boundary value problem (IBVP)

$$\begin{align}
\text{(1a)} \qquad & y_t + f(y)_x = g(\cdot, y, u_1), && \text{on } \Omega_{\bar{t}} := (0, \bar{t}) \times (\mathfrak{a}, \mathfrak{b}), \\
\text{(1b)} \qquad & y(0, \cdot) = u_0(\cdot; w), && \text{on } \Omega := (\mathfrak{a}, \mathfrak{b}), \\
\text{(1c)} \qquad & y(\cdot, \mathfrak{a}+) = u_{B,\mathfrak{a}}(\cdot; w), && \text{in the sense of (6a)}, \\
\text{(1d)} \qquad & y(\cdot, \mathfrak{b}-) = u_{B,\mathfrak{b}}(\cdot; w), && \text{in the sense of (6b)},
\end{align}$$

where $-\infty < \mathfrak{a} < a < b < \mathfrak{b} < \infty$, and the following state constraints are satisfied

$$\text{(2)} \qquad y(\bar{t}, x) \leq \bar{y}(x) \quad \text{for all } x \in [a, b].$$

Here, we associate with the control $w = (u^0, u^{B,\mathfrak{a}}, u^{B,\mathfrak{b}}, x^0, t^\mathfrak{a}, t^\mathfrak{b}, u_1) \in W_{ad}$, where $W_{ad} \subset C^1(\overline{\Omega})^{n_x+1} \times C^1([0, \bar{t}])^{n_{t,\mathfrak{a}}+1} \times C^1([0, \bar{t}])^{n_{t,\mathfrak{b}}+1} \times \mathbb{R}^{n_x} \times [0, \bar{t}]^{n_{t,\mathfrak{a}}} \times [0, \bar{t}]^{n_{t,\mathfrak{b}}} \times C([0, \bar{t}]; C^1(\mathbb{R})^m)$, the initial and boundary data

$$\text{(3)} \qquad u_0(x; w) = \begin{cases} u_1^0(x) & \text{if } x \in (\mathfrak{a}, x_1^0], \\ u_j^0(x) & \text{if } x \in (x_{j-1}^0, x_j^0], 2 \leq j \leq n_x, \\ u_{n_x+1}^0(x) & \text{if } x \in (x_{n_x}^0, \mathfrak{b}), \end{cases}$$

$$(4) \qquad u_{B,\mathfrak{a}/\mathfrak{b}}(t; w) = \begin{cases} u_1^{B,\mathfrak{a}/\mathfrak{b}}(t) & \text{if } t \in [0, t_1^{\mathfrak{a}/\mathfrak{b}}], \\ u_j^{B,\mathfrak{a}/\mathfrak{b}}(t) & \text{if } t \in (t_{j-1}^{\mathfrak{a}/\mathfrak{b}}, t_j^{\mathfrak{a}/\mathfrak{b}}], 2 \le j \le n_{t,\mathfrak{a}/\mathfrak{b}}, \\ u_{n_{t,\mathfrak{a}/\mathfrak{b}}+1}^{B,\mathfrak{a}/\mathfrak{b}}(t) & \text{if } t \in (t_{n_{t,\mathfrak{a}/\mathfrak{b}}}, \bar{t}]. \end{cases}$$

Since hyperbolic conservation laws do not admit unique weak solutions (see [1]), one has to consider entropy solutions of (1) in the sense of [17] in order to guarantee uniqueness, see (5) below. Optimal control problems with state constraints have been studied in several papers, e.g., [6, 12, 14, 16]. But to the best of the authors knowledge the optimal control of hyperbolic balance laws with state constraints has not been discussed so far.

The derivation of optimality conditions for the optimal control of hyperbolic balance laws with pointwise state constraints is involved, since in order to guarantee a constraint qualification one has usually to assume that the control-to-state mapping is continuously differentiable to L^∞. Since it is well known that entropy solutions develop shocks after finite time (see e.g. [1]), the control-to-state mapping is in this case not even continuous to L^∞.

Optimal control of hyperbolic balance laws has been considered in several papers, e.g. [4, 5, 7, 18, 19, 24, 25, 27, 28]. The developments in this paper are mainly based on [24], where the concept of shift-differentiability [27, 28] has been extended to initial-boundary value problems of the form (1). The proofs of the results in [24, 25, 28, 29] are based on the special structure of solutions of hyperbolic balance laws, that was derived by using the concept of generalized characteristic in [9]. In this paper we want to exploit this structural properties and introduce new state variables with C^1 regularity as a technical tool. This will allow us to derive first order necessary optimality conditions for (P) in terms of the new state variables. These optimality conditions can be transformed back to the original formulation of problem (P). Furthermore, using a Moreau-Yosida regularization approach to handle state constraints algorithmically, which was first introduced in [14], we can show that the optimal control and state of the regularized problems converge strongly to the optimal control and state of problem (P) (see e.g. [10, 21]), while the correpsonding Lagrange multiplier approximations obtained form the regularized problem converge weakly to the Lagrange multipliers of (P) for the state constraints.

The paper is organized as follows. In section 2 we introduce basic assumptions and collect results concerning the well-posedness of the IBVP and structural properties of its solution. In section 3 we discuss the optimal control problem (P), prove existence of a global optimum and derive first order necessary optimality conditions. In section 4 we apply the Moreau-Yosida regularization approach and prove convergence to the solution of the optimal control problem with state constraints.

2. The initial-boundary value problem. In this section we collect some results for the IBVP (1). The norm of $L^r(D)$ with a measurable domain D will be denoted by $\| \cdot \|_{r,D}$, $1 \le r \le \infty$, and the scalar product of $L^2(D)$ by $(\cdot, \cdot)_{2,D}$.

2.1. Notion of a solution for the IBVP and basic assumptions. We consider entropy solutions of (1) in the sense of [17], i.e., for every (Kružkov-) entropy $\eta_c(\lambda) := |\lambda - c|$, $c \in \mathbb{R}$, and associated entropy flux $q_c(\lambda) := \text{sgn}(\lambda - c)(f(\lambda) - f(c)) y$ has to satisfy

$$(5a) \qquad (\eta_c(y))_t + (q_c(y))_x - \eta_c'(y)g(\cdot, y, u_1) \le 0 \qquad \text{in } \mathcal{D}'(\Omega_{\bar{t}}),$$

$$(5b) \qquad \underset{t \to 0+}{\text{esslim}} \, \|y(t, \cdot) - u_0\|_{1, \Omega \cap (-R, R)} = 0 \qquad \text{for all } R > 0.$$

In order to guarantee that the problem is well-posed, the boundary conditions in (1) have to be understood in the sense of [1], i.e.

(6a) $\quad \min_{k \in I(y(\cdot,\mathfrak{a}+),u_{B,\mathfrak{a}})} \operatorname{sgn}(u_{B,\mathfrak{a}} - y(\cdot,\mathfrak{a}+))(f(y(\cdot,\mathfrak{a}+)) - f(k)) = 0 \quad$ a.e. on $[0,\bar{t}]$,

(6b) $\quad \min_{k \in I(y(\cdot,\mathfrak{b}-),u_{B,\mathfrak{b}})} \operatorname{sgn}(y(\cdot,\mathfrak{b}-) - u_{B,\mathfrak{b}})(f(y(\cdot,\mathfrak{b}-)) - f(k)) = 0 \quad$ a.e. on $[0,\bar{t}]$,

with $I(\alpha,\beta) := [\min(\alpha,\beta), \max(\alpha,\beta)]$. The condition (6) involves boundary traces $y(\cdot,\mathfrak{a}+)$ and $y(\cdot,\mathfrak{b}-)$. These limits exist if e.g. $y(t,\cdot) \in BV(\mathfrak{a},\mathfrak{b})$, which holds under suitable assumptions, see Proposition 2. We will work under the following assumptions.

(A1) The flux function satisfies $f \in C^3_{\mathrm{loc}}(\mathbb{R})$, $f'^{-1} \in C^{2,\beta}_{\mathrm{loc}}(\mathbb{R})$ for some $\beta \in (0,1]$ and is strongly convex, i.e., there exists a positive constant $m_{f''} > 0$ such that $f'' \geq m_{f''}$. Moreover, we assume that $g \in C\left([0,\bar{t}]; C^1_{\mathrm{loc}}(\mathbb{R} \times \mathbb{R} \times \mathbb{R}^m)\right)$ and that there exists $\varepsilon_g > 0$ such that for all $(y,u_1) \in \mathbb{R} \times \mathbb{R}^m$

$$g(\cdot,y,u_1) = 0 \quad \text{on } [0,\varepsilon_g] \times \mathbb{R} \cup [0,\bar{t}] \times \mathbb{R} \setminus (\mathfrak{a} + \varepsilon_g, \mathfrak{b} - \varepsilon_g).$$

Finally, let g be Lipschitz w.r.t. x and affine linear w.r.t. y.

(A2) The set of admissible controls W_{ad} is nonempty, convex and compact in

$$W := C^1(\overline{\Omega})^{n_x+1} \times U^\alpha_{B,\mathfrak{a}} \times U^\alpha_{B,\mathfrak{b}} \times \mathcal{X} \times \mathcal{T}^\mathfrak{a} \times \mathcal{T}^\mathfrak{b} \times C([0,\bar{t}]; C^1(\mathbb{R})^m),$$

where $\mathcal{X} := \left\{ \vec{x} \in \Omega^{n_x} : \mathfrak{a} < x_1 < \ldots < x_{n_x} < \mathfrak{b} \right\}$

$\mathcal{T}^\mathfrak{a} := \left\{ \vec{t} \in [0,\bar{t}]^{n_{t,\mathfrak{a}}} : 0 < t_1 < \ldots < t_{n_{t,\mathfrak{a}}} < \bar{t} \right\}$

$\mathcal{T}^\mathfrak{b} := \left\{ \vec{t} \in [0,\bar{t}]^{n_{t,\mathfrak{b}}} : 0 < t_1 < \ldots < t_{n_{t,\mathfrak{b}}} < \bar{t} \right\}$

$U^\alpha_{B,\mathfrak{a}} := \left\{ u^{B,\mathfrak{a}} \in C^1([0,\bar{t}])^{n_{t,\mathfrak{a}}+1} : f'(u^{B,\mathfrak{a}}_j) \geq \alpha, j = 1, \ldots, n_{t,\mathfrak{a}}+1 \right\}$

$U^\alpha_{B,\mathfrak{b}} := \left\{ u^{B,\mathfrak{b}} \in C^1([0,\bar{t}])^{n_{t,\mathfrak{b}}+1} : f'(u^{B,\mathfrak{b}}_j) \leq -\alpha, j = 1, \ldots, n_{t,\mathfrak{b}}+1 \right\}$

for some $\alpha > 0$ and W is equipped with the norm

$$\|w\|_W := \|u^0\|_{C^1(\overline{\Omega})^{n_x+1}} + \|u^{B,\mathfrak{a}}\|_{C^1([0,\bar{t}])^{n_{t,\mathfrak{a}}+1}} + \|u^{B,\mathfrak{b}}\|_{C^1([0,\bar{t}])^{n_{t,\mathfrak{b}}+1}}$$
$$+ \|x^0\|_1 + \|t^\mathfrak{a}\|_1 + \|t^\mathfrak{b}\|_1 + \|u_1\|_{C([0,t];C^1(\mathbb{R})^m)}$$

For all $w \in W_{ad}$, it holds that $\|u_1\|_{L^\infty(0,\bar{t};C^1(\overline{\Omega})^m)} \leq M_u$ for some $M_u > 0$.

NOTATION 1. *Given some $w \in W$, we will use the abbreviations $[\psi(x)] := \psi(x-) - \psi(x+)$, $I^j_0(w) := (x^0_{j-1}, x^0_j)$ for $j = 1, \ldots, n_x+1$ and $I^j_{B,\mathfrak{a}/\mathfrak{b}}(w) := (t^{\mathfrak{a}/\mathfrak{b}}_{j-1}, t^{\mathfrak{a}/\mathfrak{b}}_j)$ for $j = 1, \ldots, n_{t,\mathfrak{a}/\mathfrak{b}}+1$ and set $x^0_0 := \mathfrak{a}, x^0_{n_x+1} := \mathfrak{b}, t^{\mathfrak{a}/\mathfrak{b}}_0 = 0$ and $t^{\mathfrak{a}/\mathfrak{b}}_{n_{t,\mathfrak{a}/\mathfrak{b}}+1} = \bar{t}$. Furthermore, we define $B^W_\rho(w) := \{\hat{w} \in W : \|w - \hat{w}\|_W \leq \rho\}$ and the sets of indices*

$I_{s,0}(w) := \left\{ j \in \{1, \ldots, n_x\} : [u_0(x^0_j)] \geq 0 \right\}$,

$I_{s,\mathfrak{a}}(w) := \left\{ j \in \{1, \ldots, n_{t,\mathfrak{a}}\} : [u_{B,\mathfrak{a}}(t^\mathfrak{a}_j)] \leq 0 \right\}$,

$I_{s,\mathfrak{b}}(w) := \left\{ j \in \{1, \ldots, n_{t,\mathfrak{b}}\} : [u_{B,\mathfrak{b}}(t^\mathfrak{b}_j)] \geq 0 \right\}$,

$I_{r,0}(w) := \{1, \ldots, n_x\} \setminus I_{s,0}(w), I_{r,\mathfrak{a}/\mathfrak{b}}(w) := \{1, \ldots, n_{\mathfrak{a}/\mathfrak{b}}\} \setminus I_{s,\mathfrak{a}/\mathfrak{b}}(w)$.

2.2. Existence and uniqueness of a solution for the IBVP. We obtain the following existence and uniqueness result for entropy solution of (1) [1,8,22]. Oleinik's entropy condition (7) can be found in [23, Lemma 3.1.13].

PROPOSITION 2. *Let* (A1) *and* (A2) *hold. Then for every* $w \in W$ (1) *has a unique entropy solution* $y = y(w) \in L^\infty(\Omega_{\bar{t}})$. *Moreover, after a modification on a set of measure zero* y *satisfies* $y \in C([0, \bar{t}]; L^1_{\text{loc}}(\Omega))$ *and there exist constants* $M_y, L_y > 0$ *such that for all* $w, \hat{w} \in W_{\text{ad}}$ *and all* $t \in [0, \bar{t}]$ *the following estimates hold.*

$$\|y(t, \cdot; w)\|_{\infty, [\mathfrak{a}, \mathfrak{b}]} \leq M_y,$$

$$\|y(t, \cdot; w) - y(t, \cdot; \hat{w})\|_{1, [a, b]} \leq L_y \big(\|u_0(w) - u_0(\hat{w})\|_{1, I_t} + \|u_{B, \mathfrak{a}}(w) - u_{B, \mathfrak{a}}(\hat{w})\|_{1, [0, t]}$$
$$+ \|u_{B, \mathfrak{b}}(w) - u_{B, \mathfrak{b}}(\hat{w})\|_{1, [0, t]} + \|u_1 - \hat{u}_1\|_{1, [0, t] \times I_t} \big),$$

where $a < b$, $I_t := [a - t M_{f'}, b + t M_{f'}] \cap \Omega$ *and* $M_{f'} := \max_{|y| \leq M_y} |f'(y)|$. *Finally,* y *satisfies* $y \in BV(\Omega_{\bar{t}})$ *and there exists a constant* $C > 0$ *such that for all* $t \in (0, \bar{t}]$ *and all* $\varepsilon_1, \varepsilon_2 > 0$ *the solution* y *satisfies*

$$(7) \qquad y_x(t, \cdot) \leq \frac{C}{1 - e^{-m_{f''} C \min(t, C\varepsilon_1, C\varepsilon_2)}} \quad \text{on } [\mathfrak{a} + \varepsilon_1, \mathfrak{b} - \varepsilon_2]$$

in the sense of distributions yielding $y(t, x-) \geq y(t, x+)$ *for all* $(t, x) \in \Omega_{\bar{t}}$.

CONVENTION 3. *We consider the representative of* y *satisfying* $y \in C([0, \bar{t}]; L^1(\Omega))$ *and* $y(t, x) = y(t, x-)$ *for all* $(t, x) \in [0, \bar{t}] \times (\mathfrak{a}, \mathfrak{b}]$, $y(t, \mathfrak{a}) = y(t, \mathfrak{a}+)$ *for all* $t \in [0, \bar{t}]$.

LEMMA 4. *Under Assumptions* (A1) *and* (A2) *for all* $w, \hat{w} \in W_{\text{ad}}$ *it holds for* $u_0, u_{B, \mathfrak{a}}, u_{B, \mathfrak{b}}$ *defined in* (3) *and* (4) *that*

$$\|u_0(w) - u_0(\hat{w})\|_{1, [\mathfrak{a}, \mathfrak{b}]} \leq (|\mathfrak{b} - \mathfrak{a}| + \|w\|_W + \|\hat{w}\|_W) \|w - \hat{w}\|_W$$
$$\|u_{B, \mathfrak{a}/\mathfrak{b}}(w) - u_{B, \mathfrak{a}/\mathfrak{b}}(\hat{w})\|_{1, [0, \bar{t}]} \leq (\bar{t} + \|w\|_W + \|\hat{w}\|_W) \|w - \hat{w}\|_W.$$

Proof. Since by (3) the locations of the ith discontinuity differ by $|x_i^0 - \hat{x}_i^0|$ and $|u_0(x; w) - u_0(x, \hat{w})|$ can be bounded by $\|u^0\|_{C(\bar{\Omega})^{n_x+1}} + \|\hat{u}^0\|_{C(\bar{\Omega})^{n_x+1}}$, we obtain

$$\|u_0(w) - u_0(\hat{w})\|_{1, [\mathfrak{a}, \mathfrak{b}]}$$
$$\leq |\mathfrak{b} - \mathfrak{a}| \|u^0 - \hat{u}^0\|_{C(\bar{\Omega})^{n_x+1}} + (\|u^0\|_{C(\bar{\Omega})^{n_x+1}} + \|\hat{u}^0\|_{C(\bar{\Omega})^{n_x+1}}) \|x^0 - \hat{x}^0\|_1$$
$$\leq (|\mathfrak{b} - \mathfrak{a}| + \|w\|_W + \|\hat{w}\|_W) \|w - \hat{w}\|_W.$$

Analogously, one can show the remaining two inequalities.   □

Using Proposition 2, Lemma 4 and the boundedness of W_{ad} we obtain the following corollary.

COROLLARY 5. *Let* (A1) *and* (A2) *hold. Then there exists* $L_y > 0$ *such that*

$$\|y(t, \cdot; w) - y(t, \cdot; \hat{w})\|_{1, [a, b]} \leq L_y \|w - \hat{w}\|_W \quad \text{for all } w, \hat{w} \in W_{ad}.$$

Corollary 5, $\psi \in C^{1,1}_{\text{loc}}(\mathbb{R}^2)$ and $y_d \in BV([a, b])$ yield (cf. [29]):

COROLLARY 6. *Let* (A1) *and* (A2) *hold. Then there exists* $L > 0$ *such that*

$$\big| J(y(w)) - J(y(\hat{w})) \big| \leq L \|w - \hat{w}\|_W \quad \text{for all } w, \hat{w} \in W_{ad}.$$

2.3. Structure of the solution of the IBVP. In this section we will summarize some results of [24] concerning the structure of entropy solutions of (1) which can be obtained by Dafermos' theory of generalized characteristics [9], see [23, 24, 28, 29].

DEFINITION 7. *A Lipschitz curve $[\alpha, \beta] \subset [0, \bar{t}] \to \Omega_{\bar{t}}, t \mapsto (t, \xi(t))$ satisfying*

$$(8) \qquad\qquad \dot{\xi}(t) \in [f'(y(t, \xi(t)+)), f'(y(t, \xi(t)-))] \quad\quad a.e. \ on \ [\alpha, \beta]$$

is called a generalized characteristic on $[\alpha, \beta]$. If the lower and upper bounds in (8) coincide for almost all $t \in [\alpha, \beta]$, then ξ is called genuine. $\xi_\pm$ is called a maximal/minimal characteristic, if it satisfies $\dot{\xi}_\pm(t) = f'(y(t, \xi(t)\pm))$.

NOTATION 8. *We define the sets*

$$\mathbb{T}^{\mathfrak{a}} := \{\theta \in [0, \bar{t}] : f'(y(\theta+, \mathfrak{a}+) > 0 \ and \ f'(y(\theta-, \mathfrak{a}+) < 0\}$$

$$\mathbb{T}^{\mathfrak{b}} := \{\theta \in [0, \bar{t}] : f'(y(\theta+, \mathfrak{b}-) < 0 \ and \ f'(y(\theta-, \mathfrak{b}-) > 0\}$$

and denote for all $\theta \in \mathbb{T}^{\mathfrak{a}} \cup \mathbb{T}^{\mathfrak{b}}$ the corresponding maximal/minimal backward characteristic through $(\theta, \mathfrak{a}/\mathfrak{b})$ by $\xi^\theta_{\mathfrak{a}/\mathfrak{b}}$ and the corresponding timepoint where the characteristic leaves the domain $\Omega_{\bar{t}}$ by $\vartheta^\theta_{\mathfrak{a}/\mathfrak{b}}$. Analogously to [24], we define the domain

$$(9) \quad D_- := \bigcup_{\theta \in \mathbb{T}^{\mathfrak{a}}} \{(t, x) : t \in]\vartheta^\theta_{\mathfrak{a}}, \theta[, x \in]\mathfrak{a}, \xi^\theta_{\mathfrak{a}}(t)[\} \cup \bigcup_{\theta \in \mathbb{T}^{\mathfrak{b}}} \{(t, x) : t \in]\vartheta^\theta_{\mathfrak{b}}, \theta[, x \in]\xi^\theta_{\mathfrak{b}}(t), \mathfrak{b}[\}.$$

If $\xi^\theta_{\mathfrak{a}/\mathfrak{b}}$ ends in a point $(\vartheta^\theta_{\mathfrak{a}/\mathfrak{b}}, \mathfrak{b}/\mathfrak{a})$, we extend D_- by adding $[0, \vartheta^\theta_{\mathfrak{a}/\mathfrak{b}}] \times \Omega$ to the right hand side of (9).

In [23, Lemma 3.1.17] the following result is shown.

LEMMA 9. *Let (A1) and (A2) hold and consider some $w \in W$ satisfying*

$$(10a) \qquad \underset{t \,:\, u_{B,\mathfrak{a}}(t;w) \neq y(t, \mathfrak{a}+; w)}{\operatorname{essinf}} |f(u_{B,\mathfrak{a}}(t; w)) - f(y(t, \mathfrak{a}+; w))| > 0$$

$$(10b) \qquad \underset{t \,:\, u_{B,\mathfrak{b}}(t;w) \neq y(t, \mathfrak{b}-; w)}{\operatorname{essinf}} |f(u_{B,\mathfrak{b}}(t; w)) - f(y(t, \mathfrak{b}-; w))| > 0.$$

Then the sets $\mathbb{T}^{\mathfrak{a}/\mathfrak{b}}$ are finite and it holds that $u_{B,\mathfrak{a}}(\theta+) > u_{B,\mathfrak{a}}(\theta-)$ for all $\theta \in \mathbb{T}^{\mathfrak{a}}$ and $u_{B,\mathfrak{b}}(\theta+) < u_{B,\mathfrak{b}}(\theta-)$ for all $\theta \in \mathbb{T}^{\mathfrak{b}}$.

We define nondegeneracy of shocks according to [24]:

DEFINITION 10. *A discontinuity $\bar{x}$ of $y(\bar{t}, \cdot; w)$ is called nondegenerated, if it is not the center of a centered compression wave (see [9, Definition 4.3]) and the corresponding minimal/maximal backward characteristic through $(\bar{x}, \bar{t})$ end in some point $(\hat{x}, 0)$, $(\mathfrak{a}, \hat{t})$ or $(\mathfrak{b}, \hat{t})$ where $u_0, u_{B,\mathfrak{a}}$ or $u_{B,\mathfrak{a}}$ is continuously differentiable, respectively, or lie in the interior of a rarefaction wave which is created either by a discontinuity of u_0 or $u_{B,\mathfrak{a}/\mathfrak{b}}$.*

We will work under the following nondegeneracy condition.

(ND) A control $w \in W$ is called nondegenerated if the following holds: There is no point $x \in \Omega$ or $t \in [0, \bar{t}]$ where $u_0(\cdot; w), u_{B,\mathfrak{a}}(\cdot; w)$ or $u_{B,\mathfrak{b}}(\cdot; w)$ is continuous, but not differentiable. The corresponding solution $y(\bar{t}, \cdot; w)$ has no shock generation points on $[a, b]$ and a finite number of discontinuities $a < x_1 < \cdots < x_N < b$ that are no shock interaction points and nondegenerated according to Definition 10. Moreover, (10) is satisfied and for all $t_j^{\mathfrak{a}/\mathfrak{b}} \in \mathbb{T}^{\mathfrak{a}/\mathfrak{b}}$ one can construct a stripe S around the maximal/minimal backward characteristic ξ through the point $(t_j^{\mathfrak{a}}, \mathfrak{a})$ (or $(t_j^{\mathfrak{b}}, \mathfrak{b})$) and a continuously differentiable local solution Y such that $y(\cdot; w + \delta w)$ coincides with $Y(\cdot; w + \delta w)$ on $S \cap \Omega \times [0, t_j^{\mathfrak{a}/\mathfrak{b}} +$

$\delta t_j^{\mathfrak{a}/\mathfrak{b}}]$ for all $\delta w \in B_\rho^W(0_W)$, where $\rho > 0$ is chosen small enough. Finally, Y satisfies $f(Y(\theta, \mathfrak{a}; w)) < f(u_{B,\mathfrak{a}}(t_j^{\mathfrak{a}}+))$ or $f(Y(\theta, \mathfrak{b}; w)) > f(u_{B,\mathfrak{b}}(t_j^{\mathfrak{b}}+))$, respectively.

We extend now the results [23, Lemma 6.3.1, Theorem 6.3.8] such that shifts of rarefaction centers are also allowed by following the ideas of [23, Chapter 8.2].

LEMMA 11 (Stability of the shock position). *Let* (A1) *and* (A2) *hold and consider some* $\bar{w} \in W$ *satisfying* (ND) *and a nondegenerated shock point* $\bar{x}$ *of* $y(\bar{t}, \cdot; \bar{w})$. *Then there exist functions* $Y_{l/r}$ *constructed around the minimal and maximal characteristics through* $(\bar{t}, \bar{x})$, *an interval* $(x_l, x_r) \ni \bar{x}$ *and a Lipschitz continuous function*

$$(11) \qquad\qquad x_s : w \in B_\rho^W(\bar{w}) \mapsto x_s(w) \in (x_l, x_r)$$

such that for all $w \in B_\rho^W(\bar{w})$ *with* $\rho > 0$ *small enough,* $y(\bar{t}, \cdot; w)$ *is given by*

$$y(\bar{t}, x; w)|_{(x_l, x_s(w))} = Y_l(\bar{t}, x; w), \quad y(\bar{t}, x; w)|_{(x_s(w), x_r)} = Y_r(\bar{t}, x; w).$$

Proof. The first case where the minimal and maximal characteristics through $(\bar{t}, \bar{x})$ (denoted by $\xi_{l/r}$) end in points where the initial or boundary data are smooth has been shown in [23, Lemma 6.3.1]. We now consider the second case where ξ_l ends in the interior of a rarefaction wave produced in $(\bar{t}_j^{\mathfrak{a}}, \mathfrak{a})$ and note that all remaining cases can be treated analogously. We choose some $\tilde{t} > \bar{t}_j^{\mathfrak{a}}$ satisfying $M_{f'}(\tilde{t} - \bar{t}_j^{\mathfrak{a}}) < \frac{\varepsilon_g}{2}$. Since g is equal to zero for all $t \in [0, \varepsilon_g]$ and $x \in [\mathfrak{a}, \mathfrak{a} + \varepsilon_g] \cup [\mathfrak{b} - \varepsilon_g, \mathfrak{b}]$, defining $I = (f'(\bar{u}_{j+1}^{B,\mathfrak{a}}(\bar{t}_j^{\mathfrak{a}})) \cdot (\tilde{t} - \bar{t}_j^{\mathfrak{a}}) + \varepsilon, f'(\bar{u}_j^{B,\mathfrak{a}}(\bar{t}_j^{\mathfrak{a}})) \cdot (\tilde{t} - \bar{t}_j^{\mathfrak{a}}) - \varepsilon)$ yields

$$\forall \varepsilon > 0 \, \exists \rho > 0 : \quad y(\tilde{t}, x; w)|_I = f'^{-1}\left(\frac{x}{\tilde{t} - t_j^{\mathfrak{a}}}\right) \quad \forall w \in B_\rho^W(\bar{w}).$$

Since the mapping $B_\rho^W(\bar{w}) \ni w \mapsto y(\tilde{t}, \cdot; w) \in C^1(I)$ is continuously differentiable and $\xi_l(\tilde{t}) \in I$ holds true for ε small enough, we can consider (1) on the truncated space-time cylinder $[\tilde{t}, \bar{t}] \times \Omega$ with initial data $y(\tilde{t}, \cdot, \bar{w})$ and proceed as in the first case. $\qquad \square$

In the next theorem we will prove Fréchet-differentiability of the shock position (11) and derive an adjoint-based representation of the gradient. Formally, for a given solution y of (1) and given end data $p^{\bar{t}}$, the corresponding adjoint equation reads

$$\begin{aligned}
& p_t + f'(y)p_x = -g_y(\cdot, y, u_1)p \ \text{ on } \Omega_{\bar{t}}, \quad p(\bar{t}, \cdot) = p^{\bar{t}}(\cdot) \ \text{ on } \Omega, \\
(12) \qquad & p(t, \mathfrak{a}+) = 0 \ \text{ on } \{t \in (0, \bar{t}) \, : \, f'(y(\tilde{t}, \mathfrak{a}+; w)) \leq 0\}, \\
& p(t, \mathfrak{b}-) = 0 \ \text{ on } \{t \in (0, \bar{t}) \, : \, f'(y(\tilde{t}, \mathfrak{b}-; w)) \geq 0\}.
\end{aligned}$$

The adjoint state associated with (12) is defined as in [23, Definition 3.3]:

DEFINITION 12. *Consider a bounded function* $p^{\bar{t}}$ *that is pointwise everywhere the limit of a sequence* $(p_n^{\bar{t}})_{n \in \mathbb{N}} \subset C^{0,1}(\Omega)$ *which is bounded in* $C(\Omega) \cap W_{loc}^{1,1}(\Omega)$. *Then the adjoint state associated with* (12) *is defined by the following requirements:*

i) For all $\bar{x} \in \Omega$ *and all generalized backward characteristics* ξ *of* y *through* $(\bar{t}, \bar{x})$ *the function* $p^\xi(t) := p(t, \xi(t))$ *is given by the characteristic equation*

$$\dot{p}^\xi(t) = -g_y(t, \xi(t), y(t, \xi(t)), u_1(t, \xi(t)))p^\xi(t), t \in (0, \bar{t}] : \xi(t) \in \Omega, \ p^\xi(\bar{t}) = p^{\bar{t}}(\bar{x}).$$

ii) $p(t, x) = 0$ *for all* $(t, x) \in D_-$.

THEOREM 13 (Differentiability of shock points). *Let* (A1) *and* (A2) *hold. Consider any* $\bar{w} \in W$ *satisfying* (ND) *and a nondegenerated shock point* $\bar{x}$ *of* $y(\bar{t}, \cdot; \bar{w})$. *Then the mapping* (11) *is continuously differentiable with derivative (cf. Notation 1)*

$$\frac{\partial}{\partial w} x_s(w) \delta w = (p, g_{u_1}(\cdot, y, u_1) \delta u_1)_{2, \Omega_{\bar{t}}}$$

$$+ \sum_{i=1}^{n_x+1} (p(0, \cdot), \delta u_i^0)_{2, I_0^i(w)} + \sum_{i \in I_{s,0}(w)} p(0, x_i^0)[u_0(x_i^0)] \delta x_i^0 - \sum_{i \in I_{r,0}(w)} p_i^{r,0} \delta x_i^0$$

$$+ \sum_{i=1}^{n_{t,\mathfrak{a}}+1} (p(\cdot, \mathfrak{a}), f'(u_i^{B,\mathfrak{a}}) \delta u_i^{B,\mathfrak{a}})_{2, I_{B,\mathfrak{a}}^i(w)} + \sum_{i \in I_{s,\mathfrak{a}}(w)} p(t_i^\mathfrak{a}, \mathfrak{a})[f(y(t_i^\mathfrak{a}, \mathfrak{a}+; w))] \delta t_i^\mathfrak{a} + \sum_{i \in I_{r,\mathfrak{a}}(w)} p_i^{r,\mathfrak{a}} \delta t_i^\mathfrak{a}$$

$$- \sum_{i=1}^{n_{t,\mathfrak{b}}+1} (p(\cdot, \mathfrak{b}), f'(u_i^{B,\mathfrak{b}}) \delta u_i^{B,\mathfrak{b}})_{2, I_{B,\mathfrak{b}}^i(w)} - \sum_{i \in I_{s,\mathfrak{b}}(w)} p(t_i^\mathfrak{b}, \mathfrak{b})[f(y(t_i^\mathfrak{b}, \mathfrak{b}-; w))] \delta t_i^\mathfrak{b} + \sum_{i \in I_{r,\mathfrak{b}}(w)} p_i^{r,\mathfrak{b}} \delta t_i^\mathfrak{b}$$

for $\delta w \in W$, *where* p *denotes the adjoint state with end data* $p(\bar{t}, \cdot) = \mathbb{1}_{\bar{x}}(\cdot) \frac{1}{[y(\bar{t}, \bar{x}; \bar{w})]}$ *and*

$$(13\text{a}) \qquad p_j^{r,0} := \int_{f'(u_j^0(x_j^0))}^{f'(u_{j+1}^0(x_j^0))} \lim_{t \searrow 0} p(t, zt + x_j^0) \frac{1}{f''(f'^{-1}(z))} \, \mathrm{d}z, \quad j \in I_{r,0}(w),$$

$$(13\text{b}) \qquad p_j^{r,\mathfrak{a}} := \int_{f'(u_{j+1}^{B,\mathfrak{a}}(t_j^\mathfrak{a}))}^{f'(u_j^{B,\mathfrak{a}}(t_j^\mathfrak{a}))} \lim_{t \searrow t_j^\mathfrak{a}} p(t, z(t - t_j^\mathfrak{a})) \frac{z}{f''(f'^{-1}(z))} \, \mathrm{d}z, \quad j \in I_{r,\mathfrak{a}}(w),$$

$$(13\text{c}) \qquad p_j^{r,\mathfrak{b}} := \int_{f'(u_j^{B,\mathfrak{b}}(t_j^\mathfrak{b}))}^{f'(u_{j+1}^{B,\mathfrak{b}}(t_j^\mathfrak{b}))} \lim_{t \searrow t_j^\mathfrak{b}} p(t, z(t - t_j^\mathfrak{b})) \frac{z}{f''(f'^{-1}(z))} \, \mathrm{d}z, \quad j \in I_{r,\mathfrak{b}}(w).$$

Proof. This result is proved in Appendix A. $\qquad\square$

From Lemma 11 and Theorem 13, we can deduce the following result.

THEOREM 14. *Let* (A1) *and* (A2) *hold and consider some* $\bar{w} \in W$ *satisfying* (ND). *Then there exist constants* $\rho, \varepsilon > 0$ *and continuously differentiable mappings*

$$(14) \qquad Y_k : (x, w) \in I_k^\varepsilon \times B_\rho^W(\bar{w}) \mapsto Y_k(\bar{t}, x; w) \in \mathbb{R}$$

$$(15) \qquad x_k : w \in B_\rho^W(\bar{w}) \mapsto x_k(w) \in \left(x_k(\bar{w}) - \frac{\varepsilon}{2}, x_k(\bar{w}) + \frac{\varepsilon}{2} \right),$$

where $I_k^\varepsilon := (x_{k-1}(\bar{w}) - \varepsilon, x_k(\bar{w}) + \varepsilon)$ *for* $k = 1, \dots, K + 1$, *and* $x_0 = a, x_{K+1} = b$. *Finally, for all* $w \in B_\rho^W(\bar{w})$ *it holds*

$$(16) \quad y(\bar{t}, x; w)|_{[a,b]} = Y_1(\bar{t}, x; w) \cdot \mathbb{1}_{[a, x_1(w)]}(x) + \sum_{k=2}^{K+1} Y_k(\bar{t}, x; w) \cdot \mathbb{1}_{(x_{k-1}(w), x_k(w)]}(x)$$

and the mappings $w \in B_\rho^W(\bar{w}) \mapsto Y_k(\bar{t}, \cdot; w) \in C(I_k^\varepsilon)$ *are continuously differentiable.*

REMARK 15. *The derivatives of the mappings* x_k *in* (15) *can be computed as follows: If* $x_k(w)$ *is a discontinuity of* $y(\bar{t}, \cdot; w)$, *its derivative can be computed according to Theorem* 13. *If* $x_k(w)$ *lies on the boundary of a rarefaction wave, the corresponding derivative can be computed by solving the linearized characteristic equation (3.36a)-(3.36c) in [23]. The mappings* (14) *are those described in Lemma* 11. *The corresponding derivatives can be computed according to [23, Lemma 6.2.1 and 6.2.7], where in the case that the minimal/maximal characteristics end in the inner of rarefaction centers, we consider the truncated IBVPs as described in the proof of Lemma* 11.

Analogously to the proof of Theorem 13, one can extend [23, Theorem 5.2.6] to the case that shifts of rarefaction centers are allowed:

THEOREM 16 (Differentiability of the tracking-type functional). *Let* (A1) *and* (A2) *hold and consider some* $\bar{w} \in W$ *satisfying* (ND). *Further assume that* y_d *is approximately continuous in a neighborhood of the discontinuities of* $y(\bar{t}, \cdot, \bar{w})$ *on* $[a, b]$. *Then the mapping* $w \in B_\rho^W(\bar{w}) \mapsto \hat{J}(w) := J(y(w)) \in \mathbb{R}$ *is for sufficiently small* ρ *continuously differentiable. Let* p *denote the adjoint state according to* Definition 12 *with end data*

$$\bar{\psi}_y(x) := \int_0^1 \psi_y \left(y(\bar{t}, x+; \bar{w})) + \tau[y(\bar{t}, x; \bar{w})], y_d(x+) + \tau[y_d(x)] \right) \, d\tau.$$

Using (13), *the derivative of* $\hat{J}(w)$ *in a direction* $\delta w \in W$ *is given by*

$$\hat{J}'(w) \cdot \delta w = R'(w)\delta w + \left(p, g_{u_1}^T(\cdot, y, u_1)\delta u_1 \right)_{2,\Omega_{\bar{t}}}$$

$$+ \sum_{i=1}^{n_x+1} (p(0, \cdot), \delta u_i^0)_{2, I_0^i(w)} + \sum_{i \in I_{s,0}(w)} p(0, x_i^0)[u_0(x_i^0)]\delta x_i^0 - \sum_{i \in I_{r,0}(w)} p_i^{r,0}\delta x_i^0$$

$$+ \sum_{i=1}^{n_{t,\mathfrak{a}}+1} (p(\cdot, \mathfrak{a}), f'(u_i^{B,\mathfrak{a}})\delta u_i^{B,\mathfrak{a}})_{2, I_{B,\mathfrak{a}}^i(w)} + \sum_{i \in I_{s,\mathfrak{a}}(w)} p(t_i^{\mathfrak{a}}, \mathfrak{a})[f(y(t_i^{\mathfrak{a}}, \mathfrak{a}+; w))]\delta t_i^{\mathfrak{a}} + \sum_{i \in I_{r,\mathfrak{a}}(w)} p_i^{r,\mathfrak{a}}\delta t_i^{\mathfrak{a}}$$

$$- \sum_{i=1}^{n_{t,\mathfrak{b}}+1} (p(\cdot, \mathfrak{b}), f'(u_i^{B,\mathfrak{b}})\delta u_i^{B,\mathfrak{b}})_{2, I_{B,\mathfrak{b}}^i(w)} - \sum_{i \in I_{s,\mathfrak{b}}(w)} p(t_i^{\mathfrak{b}}, \mathfrak{b})[f(y(t_i^{\mathfrak{b}}, \mathfrak{b}-; w))]\delta t_i^{\mathfrak{b}} + \sum_{i \in I_{r,\mathfrak{b}}(w)} p_i^{r,\mathfrak{b}}\delta t_i^{\mathfrak{b}}$$

3. The optimal control problem. In this section we will prove existence of an optimal solution and necessary optimality conditions for (P).

3.1. Existence of globally optimal solutions.

THEOREM 17. *Let* (A1) *and* (A2) *hold and assume that there exists* $\tilde{w} \in W_{ad}$ *such that* $y(\bar{t}, x, \tilde{w}) \leq \bar{y}(x)$ *is satisfied for all* $x \in [a, b]$. *Then* (P) *admits a globally optimal solution.*

Proof. We show compactness of the set $\tilde{W}_{ad} := \{w \in W_{ad} : y(\bar{t}, x; w) \leq \bar{y}(x) \ \forall x \in [a, b]\}$. Since $\tilde{W}_{ad}$ is by assumption non-empty, we can consider a sequence $(w_n)_{n \in \mathbb{N}} \subseteq \tilde{W}_{ad} \subseteq W$. Due to the compactness of W_{ad} in W, there exists a subsequence, again denoted by $(w_n)_{n \in \mathbb{N}}$, converging to some $\bar{w} \in W_{ad}$ w.r.t. $\| \cdot \|_W$. Corollary 5 implies that the sequence $(y(\bar{t}, \cdot; w_n))_{n \in \mathbb{N}}$ converges in $L^1(a, b)$ to $y(\bar{t}, \cdot, \bar{w})$ and hence there exists a subsequence, again denoted by $(y(\bar{t}, \cdot; w_n))_{n \in \mathbb{N}}$, converging pointwise almost everywhere to $y(\bar{t}, \cdot, \bar{w})$ on $[a, b]$. Therefore and since $y(\bar{t}, x; w_n) \leq \bar{y}(x)$ holds for all $x \in [a, b]$ and all $n \in \mathbb{N}$, we obtain that

(17) $$y(\bar{t}, x, \bar{w}) \leq \bar{y}(x) \quad \text{for a.a. } x \in [a, b].$$

In order to show that (17) holds for all $x \in [a, b]$, let $\hat{x} \in (a, b]$ be arbitrary. Due to (17), we can choose a sequence $(x_n)_{n \in \mathbb{N}}$ with $x_n \nearrow \hat{x}$ for $n \to \infty$ such (17) holds for all $x = x_n$. By Convention 3, we obtain

$$y(\bar{t}, \hat{x}; \bar{w}) = \lim_{n \to \infty} y(\bar{t}, x_n; \bar{w}) \leq \lim_{n \to \infty} \bar{y}(x_n) = \bar{y}(\hat{x}),$$

where the last equality holds due to the continuity of $\bar{y}(\cdot)$. Since $y(\bar{t}, \mathfrak{a}, \bar{w}) = y(\bar{t}, \mathfrak{a}+, \bar{w})$ by Convention 3, we can argue analogously with a sequence $x_n \searrow \mathfrak{a}$ satisfying (17)

and deduce $y(\bar{t}, \mathfrak{a}, \bar{w}) \leq \bar{y}(\mathfrak{a})$. Thus, (17) holds for all $x \in [a, b]$ yielding that $\bar{w} \in \tilde{W}_{ad}$. Therefore, $\tilde{W}_{ad}$ is compact. We now consider a sequence $(w_n)_{n \in \mathbb{N}} \subseteq \tilde{W}_{ad}$ satisfying

$$\hat{J}(w_n) \to \inf_{w \in \tilde{W}_{ad}} \hat{J}(w) \quad \text{for } k \to \infty.$$

Since $\tilde{W}_{ad}$ is compact, there exists a convergent subsequence, again denoted by $(w_n)_{n \in \mathbb{N}}$, with $w_n \to \bar{w} \in \tilde{W}_{ad}$. Since $\hat{J}$ is Lipschitz-continuous w.r.t. w by Corollary 6, we obtain $\hat{J}(\bar{w}) = \inf_{w \in \tilde{W}_{ad}} \hat{J}(w)$ and hence, $\bar{w}$ is a global minimum for (P). □

3.2. Reformulation of the state variable. In order to derive necessary optimality conditions we need a constraint qualification which requires due to (2) that $y(\bar{t}, \cdot; w)$ is an element of $L^\infty([a, b])$. Difficulties arise from the fact that the control-to-state-mapping is not continuous to L^∞. Theorem 14 yields that for $w \in W$ satisfying (ND), $y(\bar{t}, \cdot; w)|_{[a,b]}$ can be rewritten according to (16). Hence, we introduce $(y_1(w), \ldots, y_{K+1}(w), x_1(w), \ldots, x_K(w))$ as new state variables, where

$$(18) \qquad y_k(\lambda; w) := Y_k\big(\bar{t}, x_{k-1}(w) + \lambda(x_k(w) - x_{k-1}(w)); w\big), \quad \lambda \in [0, 1].$$

$Y_1, \ldots, Y_{K+1}$ and $x_1, \ldots, x_K$ are given according to Theorem 14 and Remark 15. In terms of the new state variables the state constraints (2) read

$$(19) \qquad y_k(\lambda; w) \leq \bar{y}\big(\bar{t}, x_{k-1}(w) + \lambda(x_k(w) - x_{k-1}(w))\big) =: \bar{y}_k(\lambda; w), \lambda \in [0, 1].$$

LEMMA 18. *Let* (A1) *and* (A2) *hold and consider some* $\bar{w} \in W$ *satisfying* (ND). *Then we have*

$$y(\bar{t}, \cdot; \bar{w}) \leq \bar{y}(\cdot) \ \ \text{on } [a, b] \iff y_k(\cdot; \bar{w}) \leq \bar{y}_k(\cdot; \bar{w}) \ \ \text{on } [0, 1] \ \forall k = 1, \ldots, K + 1$$

Proof. This is obvious by using (18), (19), the representation of $y(\bar{t}, \cdot, \bar{w})$ in (16) and Convention 3, since $\bar{y}$ is continuous, $y(\bar{t}, \cdot; w)$ is continuous on $(x_k(w), x_{k+1}(w))$, $k = 0, \ldots, K$, and admits one-sided traces. □

THEOREM 19 (Continuous differentiability of the state). *Let* (A1) *and* (A2) *hold and consider some* $\bar{w} \in W$ *satisfying* (ND). *Then the mapping*

$$(20) \quad w \in B_\rho^W(\bar{w}) \mapsto \big(y_1(w), \ldots, y_{K+1}(w), x_1(w), \ldots, x_K(w)\big) \in C\left([0, 1]\right)^{K+1} \times \mathbb{R}^K$$

is for $\rho > 0$ *small enough well-defined and continuously differentiable, where*

$$(21) \quad \begin{aligned} \frac{\partial}{\partial w} y_k(\lambda; w)\delta w &= \frac{\partial}{\partial w} Y_k(\bar{t}, \lambda x_k(w) + (1 - \lambda)x_{k-1}(w); w)\delta w \\ &+ \frac{\partial}{\partial x} Y_k(\bar{t}, \lambda x_k(w) + (1 - \lambda)x_{k-1}(w); w) \cdot \big[\lambda \frac{\partial}{\partial w} x_k(w) + (1 - \lambda)\frac{\partial}{\partial w} x_{k-1}(w)\big]\delta w \end{aligned}$$

and the derivatives of $x_k(\cdot)$ *can be computed according to* Remark 15.

Proof. This theorem is a consequence of Theorem 14 and (18). □

3.3. First order necessary optimality conditions. Our aim is to derive necessary optimality conditions for (P). To this end, we consider a general problem of the form

$$(22) \qquad \min_{z \in Z} f(z) \quad \text{subject to} \quad G(z) \in \mathcal{K}, \quad z \in \mathcal{C}$$

and recall the following result.

THEOREM 20 (Karush-Kuhn-Tucker conditions). *Consider a local solution $\bar{z}$ of* (22) *and assume that the mappings $f : Z \to \mathbb{R}$ and $G : Z \to V$ are continuously differentiable in $\bar{z} \in Z$ with Banach spaces Z and V. We further assume that $\mathcal{C} \subset Z$ is closed, convex and non-empty and $\mathcal{K} \subset V$ is a closed convex cone. Finally, we assume that Robinson's CQ is satisfied at $\bar{z} \in Z$, i.e. it holds that*

$$(23) \qquad\qquad 0 \in \operatorname{int}\left(G\left(\bar{z}\right) + G'\left(\bar{z}\right)\left(\mathcal{C} - \bar{z}\right) - \mathcal{K}\right).$$

Then there exists a Lagrange multiplier $\bar{q} \in V^*$ such that

$$G\left(\bar{z}\right) \in \mathcal{K}, \quad \bar{q} \in \mathcal{K}^\circ := \left\{q \in V^* : \langle q, v \rangle_{V^*, V} \leq 0 \quad \forall v \in \mathcal{K}\right\}$$

$$\langle \bar{q}, G\left(\bar{z}\right)\rangle_{V^*, V} = 0, \quad \bar{z} \in \mathcal{C}, \quad \left\langle f'\left(\bar{z}\right) + G'\left(\bar{z}\right)^* \bar{q}, z - \bar{z}\right\rangle_{Z^*, Z} \geq 0 \quad \forall z \in \mathcal{C},$$

where V^* denotes the dual space of V and $\mathcal{K}^\circ$ the polar cone of $\mathcal{K}$.

Proof. This result can be found e.g. in [13, 30]. $\qquad\qquad\qquad\qquad\qquad$ □

Using the new state variables (20), we can write (P) in the form (22) by setting

$$(24) \qquad\qquad Z = W, \ V = C\left([0, 1]\right)^{K+1} \times \mathbb{R}^K,$$

$$(25) \qquad\qquad G_k\left(w\right) = y_k\left(\lambda; w\right) - \bar{y}_k\left(\lambda; w\right), \quad k = 1, \dots, K + 1,$$

$$(26) \qquad\qquad G_{k+K+1}\left(w\right) = x_k(w), \quad k = 1, \dots, K,$$

$$(27) \qquad\qquad \mathcal{K} = C\left([0, 1], (-\infty, 0]\right)^{K+1} \times \mathbb{R}^K, \ \text{ and } \ \mathcal{C} = W_{ad}.$$

Hereby, V^* is due to Riesz-Radon theorem given by $V^* = \mathcal{M}\left([0, 1]\right)^{K+1} \times \mathbb{R}^K$, where $\mathcal{M}\left([0, 1]\right)$ denotes the space of bounded Radon measures on $[0, 1]$ (see e.g. [3]).

LEMMA 21. *The polar cone of $\mathcal{K}$ can be characterized as follows:*

$$(28) \qquad\qquad q \in \mathcal{K}^\circ \Longleftrightarrow q = \left(\mu_1, \dots, \mu_{K+1}, 0, \dots, 0\right),$$

where $\mu_1, \dots, \mu_{K+1} \in \mathcal{M}\left([0, 1]\right)$ are nonnegative.

We are now able to derive first order necessary optimality conditions for (P):

THEOREM 22. *Let (A1) and (A2) hold and consider a local solution $\bar{w} \in W_{ad}$ for* (P) *satisfying (ND) and* (23). *Then there exist nonnegative $\mu_1, \dots, \mu_{K+1} \in \mathcal{M}\left([0, 1]\right)$ such that*

$$(29) \quad y_k\left(\cdot, \bar{w}\right) \leq \bar{y}_k\left(\cdot, \bar{w}\right) \quad \text{on } [0, 1] \quad \forall k = 1, \dots, K + 1$$

$$(30) \quad \sum_{k=1}^{K+1} \int_{[0,1]} \left(y_k\left(\lambda, \bar{w}\right) - \bar{y}_k\left(\lambda, \bar{w}\right)\right) \mathrm{d}\mu_k\left(\lambda\right) = 0$$

$$(31) \quad \hat{J}'\left(\bar{w}\right)(w - \bar{w}) + \sum_{k=1}^{K+1} \int_{[0,1]} \frac{\partial}{\partial w}\left(y_k(\lambda, \bar{w}) - \bar{y}_k(\lambda, \bar{w})\right)(w - \bar{w})\, \mathrm{d}\mu_k(\lambda) \geq 0 \ \ \forall w \in W_{ad}.$$

Proof. Setting (24)-(27), we observe that $\bar{w}$ is due to Lemma 18 a locally optimal solution for (22). Due to Theorem 19 and (A2) the assumptions of Theorem 20 are satisfied. Hence, Theorem 20 and Lemma 21 yield the statement of the above theorem. □

LEMMA 23. *Let (A1) and (A2) hold and consider some $\bar{w} \in W_{ad}$ satisfying (ND). Then* (23) *is satisfied if and only if there exists $\tilde{w} \in W_{ad}$ such that for a constant $\varepsilon > 0$ the following assertion is valid for all $k = 1, \dots, K + 1$:*

$$(32) \qquad y_k(\lambda, \bar{w}) - \bar{y}_k(\lambda, \bar{w}) + \frac{\partial}{\partial w}\left(y_k(\lambda, \bar{w}) - \bar{y}_k(\lambda, \bar{w})\right)(\tilde{w} - \bar{w}) \leq -\varepsilon \quad \forall \lambda \in [0, 1]$$

Proof. From (23) we can directly deduce (32) and vice verca. □

Now we will reformulate the optimality conditions (29), (30) and (31) in terms of the original state $y(\bar{t}, \cdot; \bar{w})$. As a first step, we rewrite (29), (30) and (31) in terms of the mappings $Y_1, \ldots, Y_{K+1}$ which were introduced in Theorem 14. In (18) we have introduced the new state variables by using the variable transformations

$$(33) \qquad \varphi_{k;\bar{w}} : [0,1] \to \big[x_{k-1}(\bar{w}), x_k(\bar{w})\big], \ \lambda \mapsto x_{k-1}(\bar{w}) + \lambda\big(x_k(\bar{w}) - x_{k-1}(\bar{w})\big),$$

$$(34) \qquad \varphi_{k;\bar{w}}^{-1} : \big[x_{k-1}(\bar{w}), x_k(\bar{w})\big] \to [0,1], \ x \mapsto \frac{x - x_{k-1}(\bar{w})}{x_k(\bar{w}) - x_{k-1}(\bar{w})}.$$

Using (33) and (21), the optimality conditions in (29)-(31) can be written as follows

$$Y_k(\bar{t}, \varphi_{k;\bar{w}}(\cdot), \bar{w}) \le \bar{y}(\varphi_{k;\bar{w}}(\cdot)) \quad \text{on } [0,1] \quad \forall k = 1, \ldots, K+1$$

$$\sum_{k=1}^{K+1} \int_{[0,1]} \big(Y_k(\bar{t}, \varphi_{k;\bar{w}}(\lambda), \bar{w}) - \bar{y}(\varphi_{k;\bar{w}}(\lambda))\big) \mathrm{d}\mu_k(\lambda) = 0$$

$$\hat{J}'(\bar{w})(w - \bar{w}) + \sum_{k=1}^{K+1} \int_{[0,1]} \Big[\frac{\partial}{\partial x}\big(Y_k(\bar{t}, \varphi_{k;\bar{w}}(\lambda); \bar{w}) - \bar{y}(\varphi_{k;\bar{w}}(\lambda))\big)$$

$$\cdot \big(\varphi_{k,\bar{w}}^{-1}(\varphi_{k,\bar{w}}(\lambda)) \frac{\partial}{\partial w} x_k(\bar{w})(w - \bar{w}) + \big(1 - \varphi_{k,\bar{w}}^{-1}(\varphi_{k,\bar{w}}(\lambda))\big) \frac{\partial}{\partial w} x_{k-1}(w)(w - \bar{w})\big)$$

$$+ \frac{\partial}{\partial w}\big(Y_k(\bar{t}, \varphi_{k;\bar{w}}(\lambda); \bar{w}) - \bar{y}(\varphi_{k;\bar{w}}(\lambda))\big) \cdot (w - \bar{w})\Big] \mathrm{d}\mu_k(\lambda) \ge 0 \quad \forall w \in W_{ad}.$$

Using [11, V, §3, (3.1)], one can show that there exist nonnegative measures $\bar{\mu}_k \in \mathcal{M}(I_k)$, where $I_k := [x_{k-1}(\bar{w}), x_k(\bar{w})]$, which are given by $\bar{\mu}_k(A) := \mu_k(\varphi_{k;\bar{w}}^{-1}(A))$ for all measurable $A \subset I_k$ and all $k = 1, \ldots, K+1$ such that the following holds true:

$$(35) \quad Y_k(\bar{t}, \cdot, \bar{w}) \le \bar{y}(\cdot) \quad \text{on } I_k \quad \forall k = 1, \ldots, K+1$$

$$(36) \quad \sum_{k=1}^{K+1} \int_{I_k} (Y_k(\bar{t}, x, \bar{w})) - \bar{y}(x) \mathrm{d}\bar{\mu}_k(x) = 0$$

$$(37) \quad \hat{J}'(\bar{w})(w - \bar{w}) + \sum_{k=1}^{K+1} \Big[\int_{I_k} \frac{\partial}{\partial x}[Y_k(\bar{t}, x, \bar{w}) - \bar{y}(x)] \frac{x - x_{k-1}(\bar{w})}{x_k(\bar{w}) - x_{k-1}(\bar{w})} \mathrm{d}\bar{\mu}_k(x)$$

$$\cdot \frac{\partial}{\partial w} x_k(\bar{w})(w - \bar{w}) + \int_{I_k} \frac{\partial}{\partial x}[Y_k(\bar{t}, x, \bar{w}) - \bar{y}(x)] \frac{x_k(\bar{w}) - x}{x_k(\bar{w}) - x_{k-1}(\bar{w})} \mathrm{d}\bar{\mu}_k(x)$$

$$\cdot \frac{\partial}{\partial w} x_{k-1}(\bar{w})(w - \bar{w}) + \int_{I_k} \frac{\partial}{\partial w} Y_k(\bar{t}, x, \bar{w})(w - \bar{w}) \mathrm{d}\bar{\mu}_k(x)\Big] \ge 0 \quad \forall w \in W_{ad}.$$

In connection with Theorem 22, we then obtain the following result.

THEOREM 24. *Let (A1) and (A2) hold and consider a locally optimal solution $\bar{w} \in W_{ad}$ of (P) satisfying (ND) and (23). Then there exist $K+1$ nonnegative measures $\bar{\mu}_k \in \mathcal{M}(I_k)$ such that (35)-(37) hold.*

To simplify (37), we use the following observation.

LEMMA 25. *Let (A1) and (A2) hold and consider some $\bar{w} \in W_{ad}$ and nonnegative measures $\mu_k \in \mathcal{M}(I_k)$, $k = 1, \ldots, K+1$ such that (35) and (36) are satisfied. Then*

for all measurable $A \subseteq (x_{k-1}(\bar{w}), x_k(\bar{w}))$ the following holds

$$(38) \qquad \int_A \frac{\partial}{\partial x} \left[Y_k(\bar{t}, x, \bar{w}) - \bar{y}(x) \right] \left(\frac{x - x_{k-1}(\bar{w})}{x_k(\bar{w}) - x_{k-1}(\bar{w})} \right) \mathrm{d}\bar{\mu}_k(x) = 0,$$

$$(39) \qquad \int_A \frac{\partial}{\partial x} \left[Y_k(\bar{t}, x, \bar{w}) - \bar{y}(x) \right] \left(\frac{x_k(\bar{w}) - x}{x_k(\bar{w}) - x_{k-1}(\bar{w})} \right) \mathrm{d}\bar{\mu}_k(x) = 0.$$

Proof. For arbitrary $k \in \{1, \ldots, K+1\}$ and measurable $A \subseteq (x_{k-1}(\bar{w}), x_k(\bar{w}))$ we define $A_1 := \{x \in A : Y_k(\bar{t}, x, \bar{w}) < \bar{y}(x)\}$, $A_2 := \{x \in A : Y_k(\bar{t}, x, \bar{w}) = \bar{y}(x)\}$ and observe that A_1 and A_2 are both measurable due to the regularity of $Y_k(\bar{t}, \cdot; \bar{w})$ and $\bar{y}(\cdot)$. From (35) we can deduce that $A = A_1 \cup A_2$. We firstly prove that

$$(40) \qquad \bar{\mu}_k(A_1) = 0, \quad \frac{\partial}{\partial x} \left[Y_k(\bar{t}, x, \bar{w}) - \bar{y}(x) \right] \Big|_{A_2} \equiv 0.$$

To prove the first assertion, we suppose that $\bar{\mu}_k(A_1) \neq 0$. We observe that due to the nonnegativity of $\bar{\mu}_k$, it holds that $\bar{\mu}_k(A_1) > 0$. Then we obtain by (35)

$$\sum_{j=1}^{K+1} \int_{I_j} (Y_j(\bar{t}, x, \bar{w})) - \bar{y}(x) \, \mathrm{d}\bar{\mu}_j(x) \le \int_{I_k} (Y_k(\bar{t}, x, \bar{w})) - \bar{y}(x) \, \mathrm{d}\bar{\mu}_k(x)$$

$$\le \int_{I_k \cap A_1} (Y_k(\bar{t}, x, \bar{w})) - \bar{y}(x) \, \mathrm{d}\bar{\mu}_k(x) < 0.$$

This is a contradiction to (36) and hence the first assertion is proved.

To prove the second assertion, we note that by (35) the set A_2 consists of global maximizers of the differentiable function $Y_k(\bar{t}, \cdot, \bar{w}) - \bar{y}$ on the open set $(x_{k-1}(\bar{w}), x_k(\bar{w}))$. Hence, $\frac{\partial}{\partial x} \left[Y_k(\bar{t}, x, \bar{w}) - \bar{y}(x) \right] |_{A_2} \equiv 0$. This shows the second assertion.

Recalling $A = A_1 \cup A_2$, and using (40), we obtain

$$\int_A \frac{\partial}{\partial x} \left[Y_k(\bar{t}, x, \bar{w}) - \bar{y}(x) \right] \left(\frac{x - x_{k-1}(\bar{w})}{x_k(\bar{w}) - x_{k-1}(\bar{w})} \right) \mathrm{d}\bar{\mu}_k(x) = 0$$

Analogously, we can prove (39). □

Using this result, we can further simplify the optimality conditions in Theorem 24:

COROLLARY 26. *Let* (A1) *and* (A2) *hold and consider some* $\bar{w} \in W_{ad}$ *and nonnegative measures* $\mu_k \in \mathcal{M}(I_k)$, $k = 1, \ldots, K+1$, *satisfying* (35), (36) *and* (37). *Then* (37) *can also be written as:*

$$(41) \quad \hat{J}'(\bar{w})(w - \bar{w}) + \sum_{k=1}^{K+1} \left[\frac{\partial}{\partial x} [Y_k(\bar{t}, x_k(\bar{w}), \bar{w}) - \bar{y}(x_k(\bar{w}))] \cdot \bar{\mu}_k(\{x_k(\bar{w})\}) \right.$$

$$\cdot \frac{\partial}{\partial w} x_k(\bar{w})(w - \bar{w}) + \frac{\partial}{\partial x} [Y_k(\bar{t}, x_{k-1}(\bar{w}), \bar{w}) - \bar{y}(x_{k-1}(\bar{w}))] \cdot \bar{\mu}_k(\{x_{k-1}(\bar{w})\})$$

$$\left. \cdot \frac{\partial}{\partial w} x_{k-1}(\bar{w})(w - \bar{w}) + \int_{I_k} \frac{\partial}{\partial w} Y_k(\bar{t}, x, \bar{w})(w - \bar{w}) \, \mathrm{d}\bar{\mu}_k(x) \right] \ge 0, \quad \forall w \in W_{ad}$$

Using the representation of y in (16) and Convention 3, we can formulate the optimality conditions from Theorem 24 also in terms of the original state $y(\bar{t}, \cdot; \bar{w})$:

THEOREM 27. *Let* (A1) *and* (A2) *hold and let* $\bar{w} \in W_{ad}$ *be a local solution of* (P) *satisfying* (ND) *and* (23). *Then* (35)-(37) *and* (41) *are still valid, if we replace* $Y_k(\bar{t}, x; \bar{w})$, $\frac{\partial}{\partial w} Y_k(\bar{t}, x; \bar{w})$ *and* $\frac{\partial}{\partial x} Y_k(\bar{t}, x; \bar{w})$ *by* $y(\bar{t}, x; \bar{w})$, $\frac{\partial}{\partial w} y(\bar{t}, x-; \bar{w})$ *and* $\frac{\partial}{\partial x} y(\bar{t}, x-; \bar{w})$, *respectively.*

Lemma 28. *Let* (A1) *and* (A2) *hold and consider some* $\bar{w} \in W_{ad}$ *satisfying* (ND). *Then* (23) *is satisfied if and only if there exists a constant* $\varepsilon > 0$ *and* $\delta w \in W_{ad} - \bar{w}$ *such that for all* $x \in I_k$ *and* $k = 1, \ldots, K + 1$ *it holds true that*

$$(42) \quad Y_k(\bar{t}, x, \bar{w}) - \bar{y}(x) + \frac{\partial}{\partial w} Y_k(\bar{t}, x, \bar{w}) \delta w + \frac{\partial}{\partial x} [Y_k(\bar{t}, x, \bar{w}) - \bar{y}(x)] \cdot$$
$$\left(\frac{x - x_{k-1}(\bar{w})}{x_k(\bar{w}) - x_{k-1}(\bar{w})} \frac{\partial}{\partial w} x_k(\bar{w}) \delta w + \frac{x_k(\bar{w}) - x}{x_k(\bar{w}) - x_{k-1}(\bar{w})} \frac{\partial}{\partial w} x_{k-1}(\bar{w}) \delta w \right) \leq -\varepsilon$$

Proof. This result follows directly from Lemma 23 and (18). □

Theorem 29. *Let* (A1) *and* (A2) *hold and consider some* $\bar{w} \in W_{ad}$ *satisfying* (ND), (2) *and assume that* $\bar{w} + \delta w \in W_{ad}$ *holds true for all* $\delta w \in B_\rho^W(\bar{w}) \cap \{w \in W : w \leq 0 \wedge u_1 = 0\}$ (*"$w \leq 0$" has to be understood componentwise*), *if* $\rho > 0$ *is small enough. We further assume that* $g|_{\Omega_{\bar{t}}} \equiv 0$, *the upper bound* $\bar{y}(\cdot)$ *in* (2) *is a constant and* a *and* b *are points of continuity of* $y(\bar{t}, \cdot; \bar{w})$. *Then* (23) *is satisfied in* $\bar{w}$.

Proof. We will prove Theorem 29 by applying Lemma 28. More precisely, we show that there exist $\varepsilon > 0$ and $\delta w \in W_{ad} - \bar{w}$ such that for all $x \in [x_{k-1}(\bar{w}), x_k(\bar{w})]$ and $k = 1, \ldots, K+1$ (42) holds true. By assumption, $x_k(\bar{w}) \in [a, b]$ with $k \in \{1, \ldots, K+1\}$ is either a discontinuity of $y(\bar{t}, \cdot, \bar{w})$ or lies on the boundary of a rarefaction wave. Moreover, all discontinuities of $y(\bar{t}, \cdot, \bar{w})$ on $[a, b]$ are nondegenerated according to Definition 10. Therefore, we can w.l.o.g. restrict ourselves to the case $K = 2$, in which we can discuss all relevant cases that can occur. For the case $K > 2$, the subsequent procedure can just be continued.

We assume that $(\bar{t}, x_1(\bar{w}))$ lies on the right boundary of a rarefaction wave being created in a discontinuity $\bar{t}_j^{\mathfrak{a}}$ of the left boundary data $u_{B,\mathfrak{a}}(\cdot; \bar{w})$. In addition, let the genuine backward characteristic through $(\bar{t}, a)$ also end in $(\bar{t}_j^{\mathfrak{a}}, \mathfrak{a})$. We further assume that $(\bar{t}, x_2(\bar{w}))$ is a nondegenerated shock, where the minimal backward characteristic through $(\bar{t}, x_2(\bar{w}))$ ends in a continuity point $\tilde{t}^{\mathfrak{a}} \in (\bar{t}_{j-1}^{\mathfrak{a}}, \bar{t}_j^{\mathfrak{a}})$ of $u_{B,\mathfrak{a}}(\cdot; \bar{w})$ and the maximal backward characteristic ends in a continuity point $\tilde{x} \in (\bar{x}_{l-1}^0, \bar{x}_l^0)$ of the initial data $u_0(\cdot; \bar{w})$. Finally, let the genuine backward characteristic through $(\bar{t}, b)$ also end within the interval $(\bar{x}_{l-1}^0, \bar{x}_l^0)$.

The proof consists of two steps: In **Step 1**, we will derive representations for the terms $\frac{\partial}{\partial w} Y_1(\bar{t}, \cdot; \bar{w})$, $\frac{\partial}{\partial w} Y_2(\bar{t}, \cdot; \bar{w})$, $\frac{\partial}{\partial w} Y_3(\bar{t}, \cdot; \bar{w})$, $\frac{\partial}{\partial w} x_1(\bar{w})$ and $\frac{\partial}{\partial w} x_2(\bar{w})$. In **Step 2**, we will choose $\delta w \in W_{ad} - \bar{w}$ such that (42) is satisfied for all $k = 1, 2, 3$.

Step 1: Due to $g|_{\Omega_{\bar{t}}} \equiv 0$ it holds true that $Y_1(\bar{t}, x, \bar{w}) = f'^{-1}\left(\frac{x}{\bar{t} - \bar{t}_j^{\mathfrak{a}}}\right)$ yielding

$$(43) \qquad \frac{\partial}{\partial x} Y_1(\bar{t}, x, \bar{w}) \delta x = \frac{\delta x}{(\bar{t} - \bar{t}_j^{\mathfrak{a}}) \cdot f''(f'^{-1}(\frac{x}{\bar{t} - \bar{t}_j^{\mathfrak{a}}}))},$$

$$(44) \qquad \frac{\partial}{\partial w} Y_1(\bar{t}, x, \bar{w}) \delta w = \frac{x \cdot \delta t_j^{\mathfrak{a}}}{(\bar{t} - \bar{t}_j^{\mathfrak{a}})^2 f''(f'^{-1}(\frac{x}{\bar{t} - \bar{t}_j^{\mathfrak{a}}}))}.$$

Since the minimal backward characteristic through $(\bar{t}, x_2(\bar{w}))$ ends in a continuity point $\tilde{t}^{\mathfrak{a}} \in (\bar{t}_i^{\mathfrak{a}}, \bar{t}_{i+1}^{\mathfrak{a}})$ of $u_{B,\mathfrak{a}}(\cdot; \bar{w})$ and $\bar{w} \in W_{ad}$ satisfies (ND), [23, Lemma 6.2.7] yields

$$(45) \quad \frac{\partial}{\partial x} Y_2(\bar{t}, x, \bar{w}) \delta x = \frac{(\bar{u}_j^{B,\mathfrak{a}})'(\Phi(:)) \cdot \delta x}{f''(\bar{u}_j^{B,\mathfrak{a}}(\Phi(:)))(\bar{u}_j^{B,\mathfrak{a}})'(\Phi(:))(\bar{t} - \Phi(:)) - f'(\bar{u}_j^{B,\mathfrak{a}}(\Phi(:)))}$$

$$(46) \quad \frac{\partial}{\partial w} Y_2(\bar{t}, x, \bar{w}) \delta w = -\frac{f'(\bar{u}_j^{B,\mathfrak{a}}(\Phi(:))) \cdot \delta u_j^{B,\mathfrak{a}}(\Phi(:))}{f''(\bar{u}_j^{B,\mathfrak{a}}(\Phi(:)))(\bar{u}_j^{B,\mathfrak{a}})'(\Phi(:))(\bar{t} - \Phi(:)) - f'(\bar{u}_j^{B,\mathfrak{a}}(\Phi(:)))}$$

where $(:) = (\bar{t}, x, \bar{w})$, $(\bar{u}_j^{B,\mathfrak{a}})'(\cdot)$ denotes the derivative of $\bar{u}_j^{B,\mathfrak{a}}(\cdot)$ and $\Phi(\bar{t}, x, \bar{w})$ is the unique solution of the equation $x = f'(\bar{u}_j^{B,\mathfrak{a}}(\phi))(\bar{t} - \phi) + \mathfrak{a}$ w.r.t. ϕ. We observe that

$$(47) \qquad \Phi(\bar{t}, x, \bar{w}) \in [\tilde{t}^{\mathfrak{a}}, \bar{t}_j^{\mathfrak{a}}] \quad \text{for all } x \in [x_1(\bar{w}), x_2(\bar{w})].$$

From [23, Lemma 6.2.7 (i)], we can further deduce that there exists constants $\delta_0, \beta > 0$ such that for all $x \in (x_1(\bar{w}) - \delta_0, x_2(\bar{w}) + \delta_0)$ the following is satisfied:

$$(48) \qquad q_1(:) := f''(\bar{u}_j^{B,\mathfrak{a}})(\Phi(:)))(\bar{u}_j^{B,\mathfrak{a}})'(\Phi(:))(\bar{t} - \Phi(\cdot)) - f'(\bar{u}_j^{B,\mathfrak{a}}(\Phi(:))) < -\beta$$

Now, we will have a closer look at the term Y_3. According to [23, Lemma 6.2.1], since the maximal backward characteristic through $(\bar{t}, x_2(\bar{w}))$ ends in a point $\tilde{x} \in (\bar{x}_{l-1}^0, \bar{x}_l^0)$ where the initial data $u_0(\cdot; \bar{w})$ is smooth, it holds that

$$(49) \qquad \frac{\partial}{\partial x} Y_3(\bar{t}, x, \bar{w})\delta x = \frac{(\bar{u}_l^0)'(Z(:)) \cdot \delta x}{f''(\bar{u}_l^0(Z(:)))(\bar{u}_l^0)'(Z(:))\bar{t} + 1},$$

$$(50) \qquad \frac{\partial}{\partial w} Y_3(\bar{t}, x, \bar{w})\delta w = \frac{\delta u_l^0(Z(:))}{f''(\bar{u}_l^0(Z(:)))(\bar{u}_l^0)'(Z(:))\bar{t} + 1},$$

where $Z(:) = Z(\bar{t}, x, \bar{w})$ denotes the unique solution of the equation $x = f'(\bar{u}_l^0(z))\bar{t} + z$ w.r.t. z. Then for (a possibly smaller) $\delta_0 > 0$, we can deduce from [23, Lemma 6.2.1 (i)] that for all $x \in (x_2(\bar{w}) - \delta_0, b)$ it holds true that

$$(51) \qquad q_2(:) = f''(\bar{u}_l^0(Z(:)))u_l^{0'}(Z(:))\bar{t} + 1 > \beta$$

for a (possibly smaller) positive constant $\beta > 0$.

Next, we examine the term $x_1(\bar{w})$. Since $(\bar{t}, x_1(\bar{w}))$ lies on the right boundary of a rarefaction wave and the source term g is by assumption equal to zero, we obtain that $x_1(\bar{w}) = f'(\bar{u}_j^{B,\mathfrak{a}}(\bar{t}_j^{\mathfrak{a}})) \cdot (\bar{t} - \bar{t}_j^{\mathfrak{a}}) + \mathfrak{a}$ and hence

$$(52) \quad \frac{\partial}{\partial w} x_1(\bar{w})\delta w = f''(\bar{u}_j^{B,\mathfrak{a}}(\bar{t}_j^{\mathfrak{a}}))(\bar{t} - \bar{t}_j^{\mathfrak{a}})\big[\delta u_j^{B,\mathfrak{a}}(\bar{t}_j^{\mathfrak{a}}) + (\bar{u}_j^{B,\mathfrak{a}})'(\bar{t}_j^{\mathfrak{a}})\delta t_j^{\mathfrak{a}}\big] - f'(\bar{u}_j^{B,\mathfrak{a}}(\bar{t}_j^{\mathfrak{a}}))\delta t_j^{\mathfrak{a}}.$$

The derivative of the shock position $x_2(\bar{w})$ w.r.t. w is due to Theorem 13 given by

$$(53) \quad \frac{\partial}{\partial w} x_2(\bar{w})\delta w = \sum_{k=1}^{l}(p(0, \cdot), \delta u_k^0)_{2, I_0^k \cap]0, \tilde{x}[} + \sum_{k=1}^{j}(p(\cdot, \mathfrak{a}), f'(\bar{u}_k^{B,\mathfrak{a}})\delta u_k^{B,\mathfrak{a}})_{2, I_{B,\mathfrak{a}}^k \cap]0, \tilde{t}^{\mathfrak{a}}[}$$

$$+ \sum_{k \in I_{s,0}: \bar{x}_k^0 \in]0, \tilde{x}[} p(0, \bar{x}_k^0)[\bar{u}_0(\bar{x}_k^0)]\delta x_k + \sum_{k \in I_{s,\mathfrak{a}}(\bar{w}): \bar{t}_k^{\mathfrak{a}} \in]0, \tilde{t}^{\mathfrak{a}}[} p(\bar{t}_k^{\mathfrak{a}}, \mathfrak{a})[f(y(\bar{t}_k^{\mathfrak{a}}, \mathfrak{a}+; \bar{w}))]\delta t_k^{\mathfrak{a}}$$

$$- \sum_{k \in I_{r,0}: \bar{x}_k^0 \in]0, \tilde{x}[} p_k^{r,0}\delta x_k^0 + \sum_{k \in I_{r,\mathfrak{a}}(\bar{w}): \bar{t}_k^{\mathfrak{a}} \in]0, \tilde{t}^{\mathfrak{a}}[} p_k^{r,\mathfrak{a}}\delta t_k^{\mathfrak{a}},$$

where p is the adjoint state with end data $p(\bar{t}, \cdot) = \frac{1}{[y(\bar{t}, x_2(\bar{w}); \bar{w}]}$, which is given by $p(t, x)|_{\Omega_{\bar{t}} \setminus D_-} \equiv \frac{1}{[y(\bar{t}, x_2(\bar{w}); \bar{w}]} > 0$ and equal to zero on D_-.

Step 2: Our goal is to choose δw such that (42) is satisfied for all $k \in \{1, 2, 3\}$. To this end, we choose all components of δw, except $\delta u_j^{B,\mathfrak{a}}$ and δu_l^0, equal to zero. Let

$\delta u_j^{B,\mathfrak{a}}$ and δu_l^0 be given by

$$(54)\quad \delta u_j^{B,\mathfrak{a}}(t) = \begin{cases} 0 & \text{if } \bar{t}_{j-1}^{\mathfrak{a}} \le t < \tilde{t}^{\mathfrak{a}} - \rho_1 \\ \phi_1(t) & \text{if } \tilde{t}^{\mathfrak{a}} - \rho_1 \le t < \tilde{t}^{\mathfrak{a}} \\ -\varepsilon_0 & \text{if } \tilde{t}^{\mathfrak{a}} \le t < \bar{t}_j^{\mathfrak{a}} - \rho_2 \\ \phi_2(t) & \text{if } \bar{t}_j^{\mathfrak{a}} - \rho_2 \le t < \bar{t}_j^{\mathfrak{a}} \\ \frac{-\varepsilon_0}{f''(\bar{u}_j^{B,\mathfrak{a}}(\bar{t}_j^{\mathfrak{a}}))(\bar{t} - \bar{t}_j^{\mathfrak{a}})N} & \text{if } t = \bar{t}_j^{\mathfrak{a}} \end{cases} \qquad \delta u_l^0(x) = \begin{cases} 0 & \text{if } x < \tilde{x} - \rho_3 \\ \phi_3(t) & \text{if } \tilde{x} - \rho_3 \le x < \tilde{x} \\ -\varepsilon_0 & \text{if } \tilde{x} \le x \end{cases}$$

with constants $0 < \varepsilon_0 < \rho$ and $\rho_1, \rho_2, \rho_3 > 0$. Moreover, let $N \in \mathbb{N}$ be chosen such that $f''(\bar{u}_j^{B,\mathfrak{a}}(\bar{t}_j^{\mathfrak{a}}))(\bar{t} - \bar{t}_j^{\mathfrak{a}}) \cdot N > 1$ holds true. Hereby the constants ρ_1, ρ_2 and ρ_3, which will be identified later, are independent from ε_0. Finally, let ϕ_1, ϕ_2 and ϕ_3 be chosen such that $-\varepsilon_0 \le \phi_i \le 0$ and $\bar{w} + \delta w \in W_{ad}$ holds true for sufficiently small ε_0.

We consider (42) and start with $k = 1$. We note that since $f'' \ge m_{f''}$, the function $Y_1(\bar{t}, \cdot; \bar{w})$ is monotonously increasing and hence, the only point where it may touch the upper bound on the interval $[a, x_1(\bar{w})]$ is at $x_1(\bar{w})$. From (52) and (54) we obtain that $\frac{\partial}{\partial w} x_1(\bar{w}) = \frac{-\varepsilon_0}{N}$. (2), (43), (44), the fact that $\frac{\partial}{\partial w} x_1(\bar{w}) = \frac{-\varepsilon_0}{N}$ and the choice of δw yield that the left term of (42) is at $x = x_1(\bar{w})$ bounded from above by

$$(55)\quad \frac{\partial}{\partial x} Y_1(\bar{t}, x_1(\bar{w}), \bar{w}) \cdot \frac{\partial}{\partial w} x_1(\bar{w}) \delta w = \frac{-\varepsilon_0}{N(\bar{t} - \bar{t}_j^{\mathfrak{a}}) f''(f'^{-1}(\frac{x_1(\bar{w})}{\bar{t} - \bar{t}_j^{\mathfrak{a}}}))} \le \frac{-\varepsilon_0}{N\bar{t} M_{f''}} =: -\varepsilon_{11},$$

where the inequality holds due to $0 < m_{f''} \le f''(\cdot) \le M_{f''}$. (55) and the continuity of the left term of (42) w.r.t. x yield that there exists a constant $\delta_1 > 0$ such that the left term of (42) is smaller than $-\frac{\varepsilon_{11}}{2}$ for all $x \in (x_1 - \delta_1, x_1]$. Since $Y_1(\bar{t}, \cdot; \bar{w})$ is strictly monotonous increasing, for some $\varepsilon_{12} > 0$ it holds true that

$$(56)\quad Y_1(\bar{t}, x; \bar{w}) - \bar{y} \le -\varepsilon_{12} \quad \text{for all } x \in [a, x_1(\bar{w}) - \delta_1].$$

Hence, due to (43), (54) and the fact that $\frac{\partial}{\partial w} x_1(\bar{w}) = \frac{-\varepsilon_0}{N}$, the left term of (42) is on $[a, x_1(\bar{w}) - \delta_1]$ bounded from above by

$$(57)\quad -\varepsilon_{12} - \frac{\varepsilon_0}{N \cdot (\bar{t} - \bar{t}_j^{\mathfrak{a}}) \cdot f''(f'^{-1}(\frac{x}{\bar{t} - \bar{t}_j^{\mathfrak{a}}}))} \frac{x - a}{x_1(\bar{w}) - a} \le -\varepsilon_{12}.$$

Choosing $\varepsilon = \varepsilon_1 := \min\{\frac{\varepsilon_{11}}{2}, \varepsilon_{12}\}$ yields that (42) is satisfied for $k = 1$.

Considering $k = 2$, (2) yields that the left term of (42) is bounded from above by

$$(58)\quad \begin{aligned} & \frac{\partial}{\partial w} Y_2(\bar{t}, x, \bar{w}) \delta w + \frac{\partial}{\partial x} Y_2(\bar{t}, x, \bar{w}) \Big[\frac{x_2(\bar{w}) - x}{x_2(\bar{w}) - x_1(\bar{w})} \frac{\partial}{\partial w} x_1(\bar{w}) \delta w \\ & + \frac{x - x_1(\bar{w})}{x_2(\bar{w}) - x_1(\bar{w})} \frac{\partial}{\partial w} x_2(\bar{w}) \delta w \Big]. \end{aligned}$$

(45), (46) and (52) yield that (58) is in $x = x_1(\bar{w})$ equal to $\delta u_j^{B,\mathfrak{a}}(\bar{t}_j^{\mathfrak{a}}) = \frac{-\varepsilon_0}{f''(\bar{u}_j^{B,\mathfrak{a}}(\bar{t}_j^{\mathfrak{a}}))(\bar{t} - \bar{t}_j^{\mathfrak{a}})N}$. Hence, the continuity of (58) w.r.t. x yields that (58) is on $[x_1(\bar{w}), x_1(\bar{w}) + \delta]$ bounded from above by $\frac{\delta u_j^{B,\mathfrak{a}}(\bar{t}_j^{\mathfrak{a}})}{2} := -\varepsilon_{21} < 0$, where $\delta > 0$ is sufficiently small and does not depend on ε_0. We choose ρ_2 such that $\Phi(x_1(\bar{w}) + \delta) = \bar{t}_j^{\mathfrak{a}} - \rho_2$. We further choose

$$\rho_1 = \frac{[y(\bar{t}, x_2(\bar{w}); \bar{w})]}{2N \|f'(\bar{u}_j^{B,\mathfrak{a}}(\cdot))\|_{\infty, [0, \tilde{t}^{\mathfrak{a}}]}} > 0 \quad \text{and} \quad \rho_3 = \frac{[y(\bar{t}, x_2(\bar{w}); \bar{w})]}{2N} > 0.$$

Then (53), (54) and $\|\phi_i(\cdot)\|_\infty \leq \varepsilon_0$ yield that $|\frac{\partial}{\partial w}x_2(\bar{w})\delta w| \leq \frac{\varepsilon_0}{N}$. Using (46), (48) and (54), we obtain

$$\frac{\partial}{\partial w}Y_2(\bar{t},\cdot;\bar{w}) \leq \frac{-\alpha\varepsilon_0}{\|q_1(\cdot)\|_{\infty,[x_1(\bar{w})+\delta,x_2(\bar{w})]}} \quad \text{on } [x_1(\bar{w})+\delta,x_2(\bar{w})],$$

where α is the constant in (A2). Due to the boundedness of $\frac{\partial}{\partial x}Y_2(\bar{t},x,\bar{w})$ and the fact that $|\frac{\partial}{\partial w}x_1(\bar{w})\delta w|,|\frac{\partial}{\partial w}x_2(\bar{w})\delta w| \leq \frac{\varepsilon_0}{N}$, one can choose N large enough such that (58) is on $[x_1(\bar{w})+\delta,x_2(\bar{w})]$ bounded from above by $-\varepsilon_{22} := \frac{-\alpha\varepsilon_0}{2\|q_1(\bar{t},\cdot,\bar{w})\|_{\infty,[x_1(\bar{w})+\delta,x_2(\bar{w})]}}$. Choosing $\varepsilon = \min\{\varepsilon_{21},\varepsilon_{22}\}$, (42) is satisfied for $k=2$.

Finally, we consider the case $k=3$. Using (50), (51) and (54), we obtain

(59) $$\frac{\partial}{\partial w}Y_3(\bar{t},\cdot;\bar{w}) \leq \frac{-\varepsilon_0}{\|q_2(\bar{t},\cdot,\bar{w})\|_{\infty,[x_2(\bar{w}),b]}} \quad \text{on } [x_2(\bar{w}),b].$$

Due to the boundedness of $\frac{\partial}{\partial x}Y_2(\bar{t},x,\bar{w})$ and the fact that $|\frac{\partial}{\partial w}x_2(\bar{w})\delta w| \leq \frac{\varepsilon_0}{N}$, one can choose N large enough such that (2) and (59) yield that the left term of (42) is on $[x_2(\bar{w}),b]$ smaller than $-\frac{\varepsilon_0}{2\|q_2(\bar{t},\cdot,\bar{w})\|_{\infty,[x_2(\bar{w}),b]}} =: -\varepsilon_3$. Hereby, ε_3 is finite since $q_2((\bar{t},\cdot,\bar{w})$ is continuous on the the compact interval $[x_2(\bar{w}),b]$. Hence, (55) is satisfied for $k=3$ if we choose $\varepsilon = \varepsilon_3$. Choosing $\varepsilon := \min\{\varepsilon_1,\varepsilon_2,\varepsilon_3\}$, (55) is satisfied for $k=1,2,3$ for the choice of δw in (54). $\qquad\square$

REMARK 30. *Carefully studying the results of [23, Lemma 6.2.1, Lemma 6.2.7], Theorem 29 also hold for certain source terms which are not equal to zero on $\Omega_{\bar{t}}$, e.g. for source terms which only depend on the state y and satisfy* (A1). *Furthermore, using the same arguments as in the proof above, one can show that Theorem 29 is still valid if the upper bound $\bar{y}$ is not constant.*

4. Moreau-Yosida Regularization. Since it is quite involved to compute a solution of the optimality system in Theorem 24, we will omit the state constraints and take them into account by adding a penalty function $P(y(w))$ to the cost functional, which we multiply by a penalty parameter $\frac{1}{\gamma}$, where $\gamma > 0$, and obtain

(P_γ) $$\min_{w\in W} J_\gamma(y(w)) := J(y(w)) + \frac{1}{2\gamma}\int_a^b (y(\bar{t},x;w)-\bar{y}(x))_+^2 \, dx$$

s.t. $w \in W_{ad}$ and $y(w)$ solves the (IBVP),

where $(\cdot)_+ := \max\{\cdot,0\}$. This approach is called Moreau-Yosida regularization, see for example [14], [21]. Other approaches can be found e.g. in [16], [15]. Since $J_\gamma(\cdot)$ can be written in the form of $J(\cdot)$ in (P), we obtain the following result.

THEOREM 31. *Let (A1) and (A2) hold and consider some $\bar{w} \in W$ satisfying* (ND). *Then there exists a neighborhood $B_\rho^W(\bar{w})$ of $\bar{w}$ such that the mapping $B_\rho^W(\bar{w}) \ni w \mapsto J_\gamma(y(w)) \in \mathbb{R}$ is continuously differentiable. The derivative in a direction $\delta w \in W$ can be computed according to Theorem 16.*

THEOREM 32. *Let (A1) and (A2) hold. Then for each penalty parameter γ there exists a globally optimal solution $w_\gamma \in W_{ad}$ for (P_γ).*

Proof. The proof is similar to the proof of Theorem 17. $\qquad\square$

THEOREM 33. *Let (A1) and (A2) hold and consider a sequence $(w_{\gamma_k})_{k\in\mathbb{N}} \subset W_{ad}$ of global solutions for (P_γ) with $\lim_{k\to\infty}\gamma_k = 0$. Further assume that $x = a$ is a point of continuity of $y(\bar{t},\cdot;w)$ for all $w \in W_{ad}$. Then there exists a subsequence converging strongly to some $w^* \in W_{ad}$, which is a global solution for (P).*

Proof. We consider a sequence $(w_{\gamma_k})_{k\in\mathbb{N}}$ of global optima for (P_γ) and prove that there exists a subsequence converging to a globally optimal solution for (P). Since the set $W_{ad} \subset W$ is compact, there exists a convergent subsequence (again denoted by $(w_{\gamma_k})_{k\in\mathbb{N}}$) such that $w_{\gamma_k} \to w^* \in W_{ad}$ w.r.t. $\|\cdot\|_W$. In the next step, we prove that $y(w^*)$ fulfills the state constraints. To this end, we firstly note that

$$(60) \qquad\qquad J_\gamma(y(w_k)) \leq J(y(\bar{w})) \ \text{ for all } k \in \mathbb{N},$$

where $\bar{w}$ denotes the globally optimal solution for (P), which exists due to Theorem 17. Due to (60), the continuity of $J(y(\cdot))$ w.r.t. w and the fact that $w_{\gamma_k} \to w^* \in W_{ad}$, there exists a constant $C > 0$ such that for all $k \in \mathbb{N}$ the following holds true:

$$0 \leq \frac{1}{2\gamma} \int_a^b (y(\bar{t},x;w_{\gamma_k}) - \bar{y}(x))_+^2 \ \mathrm{d}x \leq J(y(\bar{w})) - J(y(w_{\gamma_k})) \leq C$$

$$(61) \qquad \Longleftrightarrow 0 \leq \int_a^b (y(\bar{t},x;w_{\gamma_k}) - \bar{y}(x))_+^2 \ \mathrm{d}x \leq 2 \cdot \gamma_k \cdot C$$

Using (61), we prove that $y(\bar{t},x;w^*) \leq \bar{y}(x)$ holds true for all $x \in [a,b]$. From (61) and the fact that $\gamma_k \to 0$, we can deduce that $(y(\bar{t},\cdot;w_{\gamma_k}) - \bar{y}(\cdot))_+ \to 0$ in $L^2([a,b])$ and hence pointwise almost everywhere on $[a,b]$ for a subsequence (again denoted by $(w_{\gamma_k})_{k\in\mathbb{N}}$). Considering this subsequence, we know by Corollary 5 that $y(\bar{t},\cdot;w_{\gamma_k}) \to y(\bar{t},\cdot;w^*)$ in $L^1([a,b])$ and hence pointwise almost everywhere on $[a,b]$ for another subsequence, which is again denoted by $(w_{\gamma_k})_{k\in\mathbb{N}}$. Since $(y(\bar{t},\cdot;w_{\gamma_k}) - \bar{y}(\cdot))_+ \to 0$ and $y(\bar{t},\cdot;w_{\gamma_k}) \to y(\bar{t},\cdot;w^*)$ for almost all $x \in [a,b]$, we obtain:

$$\begin{aligned}
\big(y(\bar{t},x;w^*) - \bar{y}(x)\big)_+ &= \big(y(\bar{t},x;w^*) - y(\bar{t},x;w_{\gamma_k}) + y(\bar{t},x;w_{\gamma_k}) - \bar{y}(x)\big)_+ \\
&\leq \big(|y(\bar{t},x;w^*) - y(\bar{t},x;w_{\gamma_k})| + y(\bar{t},x;w_{\gamma_k}) - \bar{y}(x)\big)_+ \\
&\leq |y(\bar{t},x;w^*) - y(\bar{t},x;w_{\gamma_k})| + \big(y(\bar{t},x;w_{\gamma_k}) - \bar{y}(x)\big)_+ \to 0, \quad \text{a.e. on } [a,b]
\end{aligned}$$

and hence $(y(\bar{t},\cdot;w^*) - \bar{y}(\cdot))_+ = 0$ holds pointwise almost everywhere on $[a,b]$ which is equivalent to

$$(62) \qquad\qquad y(\bar{t},x;w^*) \leq \bar{y}(x) \quad \text{ for a.a. } x \in [a,b].$$

Analogously to the proof of Theorem 17, one can show that (62) holds true for all $x \in [a,b]$ and hence $y(\bar{t},\cdot;w^*)$ fulfills the state constraints. Furthermore, since the inequality $J_{\gamma_k}(y(w_{\gamma_k})) \leq J(y(\bar{w}))$ holds true for all $k \in \mathbb{N}$, we obtain $J(y(w^*)) \leq J(y(\bar{w}))$. Since $y(\bar{t},\cdot,w^*)$ fulfills the state constaints and $\bar{w}$ is a globally optimal solution for (P), it holds true that $J(y(w^*)) = J(y(\bar{w}))$ and w^* is hence a globally optimal solution for (P). $\qquad\qquad\square$

We will now examine the convergence of local solutions of (P_γ). To this end, we introduce for some constant $r > 0$ the auxiliary problems

$$(P^r) \qquad \min_{w\in W} J(y(w)) \ \text{ s.t. } \ y(w) \text{ solves the (IBVP)}, \ w \in W^r := W_{ad} \cap B_r^W(\bar{w})$$

$$y(\bar{t},x;w) \leq \bar{y}(x) \quad \forall x \in [a,b],$$

$$(P_\gamma^r) \qquad \min_{w\in W} J_\gamma(y(w)) \ \text{ s.t. } \ y(w) \text{ solves the (IBVP)}, \ w \in W^r$$

(cf. [6]). Similar to the proof of Theorem 17, one can show that these problems admit global solutions denoted by $\bar{w}^r$ and w_γ^r, respectively.

THEOREM 34. *Suppose that* (A1) *and* (A2) *hold and let* $\bar{w} \in W_{ad}$ *be a local optimum for* (P) *such that for constants* $\varepsilon, \delta > 0$ *the quadratic growth condition*

$$(63) \qquad J(y(\bar{w})) + \frac{\delta}{2}\|w - \bar{w}\|_H \leq J(y(w)) \quad \forall w \in \tilde{W}_{ad} \text{ with } \|w - \bar{w}\|_W < \varepsilon$$

is satisfied, where $\tilde{W}_{ad} := \{w \in W_{ad} : y(\bar{t}, x; w) \leq \bar{y}(x) \ \forall x \in [a, b]\}$ *and* $H \subset W$ *is a Hilbert space such that for all* $w_1, w_2 \in W$ *with* $\|w_1 - w_2\|_H = 0$ *it holds true that* $\|w_1 - w_2\|_W = 0$. *Further assume that* $x = a$ *is a point of continuity of* $y(\bar{t}, \cdot; w)$ *for all* $w \in W_{ad}$. *Then there exists a sequence* $(w_{\gamma_k})_{k \in \mathbb{N}}$ *of local solutions for* (P_γ) *converging to* $\bar{w}$ *w.r.t.* $\|\cdot\|_W$.

Proof. We consider a sequence of globally optimal solutions for (P_γ^r) denoted by $(w_{\gamma_k})_{k \in \mathbb{N}} \subset W^r$ with $\gamma_k \to 0$ and $r = \frac{\varepsilon}{2}$. Since W^r is compact in W (see proof of Theorem 17), there is a convergent subsequence, again denoted by $(w_{\gamma_k})_{k \in \mathbb{N}}$, converging w.r.t. $\|\cdot\|_W$ to some $w^* \in W^r$. As in the proof of Theorem 33, one can show that w^* is a globally optimal solution for (P^r) and due to (63) we obtain that $w^* = \bar{w}$. Hence, for k large enough it holds true that $w_{\gamma_k}^r \in \text{int } B_r^W(\bar{w})$. This yields that $w_{\gamma_k}^r$ is a locally optimal solution for (P_γ) if k is chosen large enough (cf. [21, Proof of Theorem 5.2]).□

THEOREM 35. *Suppose that* (A1) *and* (A2) *hold and consider a sequence* $(w_{\gamma_k})_{k \in \mathbb{N}}$ *of local solutions for* (P_{γ_k}) *such that for constants* $\varepsilon, \delta > 0$ *the condition*

$$(64) \quad J_{\gamma_k}(y(w_{\gamma_k})) + \frac{\delta}{2}\|w - w_{\gamma_k}\|_H \leq J_{\gamma_k}(y(w)) \quad \forall w \in W_{ad} \text{ with } \|w - w_{\gamma_k}\|_W < \varepsilon$$

is satisfied for all $k \in \mathbb{N}$ *large enough, where* $\|\cdot\|_H$ *is defined as in* Theorem 34. *Further assume that* $x = a$ *is a point of continuity of* $y(\bar{t}, \cdot; w)$ *for all* $w \in W_{ad}$. *Then there exists a subsequence converging to some* $\bar{w} \in W_{ad}$ *w.r.t.* $\|\cdot\|_W$, *which is a local solution for* (P).

Proof. Since W_{ad} is by assumption compact, there exists a convergent subsequence $(w_{\gamma_k})_{k \in \mathbb{N}}$ with limit $\bar{w} \in W_{ad}$. We consider the corresponding problems (P^r) and (P_γ^r) with $r = \frac{\varepsilon}{2}$. Then (64) yields that w_{γ_k} is the unique globally optimal solution for $(P_{\gamma_k}^{\frac{\varepsilon}{2}})$, if k is large enough. Using this, one can analogously to the proof of Theorem 33 show that $\bar{w} \in W_{ad}$ is a globally optimal solution for $(P^{\frac{\varepsilon}{2}})$ and hence a local solution for (P). □

LEMMA 36. *Let* (A1) *and* (A2) *hold and consider a sequence* $(w_{\gamma_k})_{k \in \mathbb{N}} \subset W$ *converging to some* $w^* \in W$ *satisfying* (ND). *Then for* k *large enough* w_{γ_k} *satisfies* (ND) *and there exists* $\varepsilon > 0$ *such that for all* $j = 1, \ldots, K + 1$ *it holds true that*

$$\lim_{k \to \infty} Y_j(\bar{t}, \cdot; w_{\gamma_k}) = Y_j(\bar{t}, \cdot; w^*) \qquad \text{in } C^1\left((x_{j-1}(w^*) - \varepsilon, x_j(w^*) + \varepsilon)\right),$$

$$\lim_{k \to \infty} \frac{\partial}{\partial w} Y_j(\bar{t}, \cdot; w_{\gamma_k}) = \frac{\partial}{\partial w} Y_j(\bar{t}, \cdot; w^*) \qquad \text{in } C\left((x_{j-1}(w^*) - \varepsilon, x_j(w^*) + \varepsilon)\right).$$

Proof. The first assertion is a consequence of [23, Lemma 3.1.10]. The second one and the fact that for k large enough w_{γ_k} satisfies (ND) follows from Theorem 14. □

THEOREM 37. *Let* (A1) *and* (A2) *hold and* $w_\gamma \in W_{ad}$ *be a local solution for* (P_γ) *satisfying* (ND). *Then it holds true that*

$$(65) \qquad \qquad \frac{\partial}{\partial w} J_\gamma(y(w_\gamma)) \cdot (w - w_\gamma) \geq 0 \quad \forall w \in W_{ad}.$$

Proof. This result can be found e.g. in [13].   □

For $w \in W$ satisfying (ND), using Theorem 14 we can rewrite $J_\gamma(y(w))$ as

$$(66) \quad J_\gamma(y(w)) = J(y(w)) + \sum_{j=1}^{K+1} z_j(w) \cdot \frac{1}{2\gamma z_j(w)} \int_{x_{j-1}(w)}^{x_j(w)} (Y_j(\bar{t}, x; w) - \bar{y}(x))_+^2 \, dx,$$

where $z_j(w) := (x_j(w) - x_{j-1}(w))$. Therefore, using the abbreviation $\hat{G}_j(\bar{t}, x; w) := (Y_j(\bar{t}, x, \bar{w}) - \bar{y}(x))$, one can rewrite the derivative of $J_\gamma(y(w))$ in some $w \in W$ satisfying (ND) in a direction $\delta w \in W$ as follows:

$$\frac{\partial}{\partial w} J_\gamma(y(w))\delta w = \frac{\partial}{\partial w} J(y(w))\delta w + \sum_{j=1}^{K+1} z_j(w) \left[\frac{1}{\gamma z_j(w)} \int_{x_{j-1}(w)}^{x_j(w)} \left(\frac{\partial}{\partial w} \hat{G}_j(\bar{t}, x; w)\delta w \right. \right.$$

$$\cdot (\hat{G}_j(\bar{t}, x; w))_+ \, dx + \frac{-1}{2\gamma z_j(w)^2} \int_{x_{j-1}(w)}^{x_j(w)} (\hat{G}_j(\bar{t}, x; w))_+^2 \, dx \cdot \frac{\partial}{\partial w} z_j(w)\delta w$$

$$+ \frac{1}{2\gamma z_j(w)} \cdot \left((\hat{G}_j(\bar{t}, x_j(w); w))_+^2 \frac{\partial}{\partial w} x_j(w)\delta w - (\hat{G}_j(\bar{t}, x_{j-1}(w); w))_+^2 \right.$$

$$\left. \left. \cdot \frac{\partial}{\partial w} x_{j-1}(w)\delta w \right) \right] + \frac{1}{2\gamma z_j(w)} \int_{x_{j-1}(w)}^{x_j(w)} (\hat{G}_j(\bar{t}, x; w))_+^2 \, dx \cdot \frac{\partial}{\partial w} z_j(w)\delta w$$

Defining

$$(67) \qquad \lambda_j(x; w) := \begin{cases} \frac{(\hat{G}_j(\bar{t}, x; w))_+}{\gamma}, & \text{for } x_{j-1}(w) \le x \le x_j(w) \\ 0, & \text{for } x \in [a, b] \setminus [x_j(w), x_{j+1}(w)], \end{cases}$$

$$(68) \qquad r_j(w) := \frac{1}{2\gamma z_j(w)} \int_{x_{j-1}(w)}^{x_j(w)} (\hat{G}_j(\bar{t}, x; w))_+^2 \, dx$$

the optimality conditions of Theorem 37 can hence be written in the following way:

THEOREM 38. *Let (A1) and (A2) hold and $w_\gamma \in W_{ad}$ be a local solution for (P_γ) satisfying* (ND). *Then it holds true that*

$$\frac{\partial}{\partial w} J(y(w_\gamma))\delta w + \sum_{j=1}^{K+1} \left[\int_{x_{j-1}(w_\gamma)}^{x_j(w_\gamma)} \frac{\partial}{\partial w} Y_j(\bar{t}, x; w_\gamma)\delta w \lambda_j(x; w_\gamma) \, dx \right.$$

$$+ \int_{x_{j-1}(w_\gamma)}^{x_j(w_\gamma)} \frac{\partial}{\partial x} (Y_j(\bar{t}, x; w_\gamma) - \bar{y}(x)) \frac{x - x_{j-1}(w_\gamma)}{x_j(w_\gamma) - x_{j-1}(w_\gamma)} \lambda_j(x; w_\gamma) \, dx \frac{\partial}{\partial w} x_j(w_\gamma)\delta w$$

$$+ \int_{x_{j-1}(w_\gamma)}^{x_j(w_\gamma)} \frac{\partial}{\partial x} (Y_j(\bar{t}, x; w_\gamma) - \bar{y}(x)) \frac{x_j(w_\gamma) - x}{x_j(w_\gamma) - x_{j-1}(w_\gamma)} \lambda_j(x; w_\gamma) \, dx$$

$$(69) \quad \left. \cdot \frac{\partial}{\partial w} x_{j-1}(w_\gamma)\delta w + r_j(w_\gamma) \frac{\partial}{\partial w} (x_j(w_{\gamma_k}) - x_{j-1}(w_{\gamma_k}))\delta w \right] \ge 0 \quad \delta w \in W_{ad}.$$

LEMMA 39. *Let (A1) and (A2) hold and consider a sequence $(w_{\gamma_k})_{k \in \mathbb{N}}$ of local solutions w_{γ_k} for (P_{γ_k}) converging to a local solution $\bar{w} \in W_{ad}$ for (P). If $\bar{w}$ satisfies* (ND), *then it holds true that*

$$(70) \qquad \lim_{k \to \infty} r_j(w_{\gamma_k}) = 0 \quad \text{for all } j = 1, \dots, K+1.$$

Proof. We note that since $\bar{w}$ satisfies (ND), Lemma 36 yields that w_{γ_k} satisfies (ND) for k large enough such that (66) is valid for all $w = w_{\gamma_k}$ if k is large enough. Using (66) and the same arguments as in the proof of Theorem 33, one can show that

$$\lim_{k\to\infty} \frac{1}{2\gamma} \int_a^b (y(\bar{t}, x; w_{\gamma_k}) - \bar{y}(x))_+^2 \, dx$$

$$(71) \qquad = \lim_{k\to\infty} \sum_{j=1}^{K+1} \frac{1}{2\gamma} \int_{x_{j-1}(w_{\gamma_k})}^{x_j(w_{\gamma_k})} (Y_j(\bar{t}, x; w_{\gamma_k}) - \bar{y}(x))_+^2 \, dx = 0.$$

Since $x_{j-1}(w_{\gamma_k})$ and $x_j(w_{\gamma_k})$ are uniformly bounded away from each other for $k \in \mathbb{N}$ large enough and the integrands in (71) are nonnegative, we obtain for $j = 1, \ldots, K+1$

$$\lim_{k\to\infty} r_j(w_{\gamma_k}) = \lim_{k\to\infty} \frac{1}{2\gamma z_j(w_{\gamma_k})} \int_{x_{j-1}(w_{\gamma_k})}^{x_j(w_{\gamma_k})} (Y_j(\bar{t}, x; w_{\gamma_k}) - \bar{y}(x))_+^2 \, dx = 0. \qquad \square$$

Considering a sequence of local solutions $(w_{\gamma_k})_{k\in\mathbb{N}}$ of (P_{γ_k}) satisfying (ND), we want to analize in which sense the terms in (67) converge. In the next lemma we prove that the sequences $(\lambda_j(x; w_{\gamma_k}))_{k\in\mathbb{N}}$ are uniformly bounded in L^1.

LEMMA 40. *Let (A1) and (A2) hold and consider a sequence $(w_{\gamma_k})_{k\in\mathbb{N}}$ of local solutions for (P_{γ_k}) converging to a local solution $\bar{w} \in W_{ad}$ for (P). If $\bar{w}$ satisfies (ND) and (23), then the sequences $(\lambda_j(\cdot; w_{\gamma_k}))_{k\in\mathbb{N}}$ are uniformly bounded in $L^1([a, b])$.*

Proof. We firstly note that Lemma 36 ensures that w_{γ_k} satisfies (ND) for sufficiently large k. Hence, Theorem 38 yields that (69) is satisfied in w_{γ_k} such that we can conclude that for all $w \in W_{ad}$, the following inequality holds true:

$$\sum_{j=1}^{K+1} \int_{x_{j-1}(w_{\gamma_k})}^{x_j(w_{\gamma_k})} - \left[\frac{\partial}{\partial w} Y_j(\bar{t}, x; w_{\gamma_k})(w - w_{\gamma_k}) + \frac{\partial}{\partial x}(Y_j(\bar{t}, x; w_{\gamma_k}) - \bar{y}(x)) \right.$$

$$\cdot \frac{x - x_{j-1}(w_{\gamma_k})}{x_j(w_{\gamma_k}) - x_{j-1}(w_{\gamma_k})} \frac{\partial}{\partial w} x_j(w_{\gamma_k})(w - w_{\gamma_k}) + \frac{\partial}{\partial x}(Y_j(\bar{t}, x; w_{\gamma_k}) - \bar{y}(x))$$

$$\left. \cdot \frac{x_j(w_{\gamma_k}) - x}{x_j(w_{\gamma_k}) - x_{j-1}(w_{\gamma_k})} \frac{\partial}{\partial w} x_{j-1}(w_{\gamma_k}) \cdot (w - w_{\gamma_k}) \right] \lambda_j(x; w_{\gamma_k}) \, dx$$

$$\leq \frac{\partial}{\partial w} J(y(w_{\gamma_k})) \cdot (w - w_{\gamma_k}) + \sum_{j=1}^{K+1} r_j(w_{\gamma_k}) \cdot \frac{\partial}{\partial w}(x_j(w_{\gamma_k}) - x_{j-1}(w_{\gamma_k})) \cdot (w - w_{\gamma_k}).$$

Using Theorem 16, Lemma 39 and the compactness of W_{ad}, we conclude that the expression on the right hand side of the previous inequality is uniformly bounded w.r.t. k and w. Therefore, there exists a positive constant $C > 0$ with

$$(72) \quad \sum_{j=1}^{K+1} \int_{x_{j-1}(w_{\gamma_k})}^{x_j(w_{\gamma_k})} - \left[\frac{\partial}{\partial w} Y_j(\bar{t}, x; w_{\gamma_k})(w - w_{\gamma_k}) + \frac{\partial}{\partial x}(Y_j(\bar{t}, x; w_{\gamma_k}) - \bar{y}(x)) \right.$$

$$\cdot \frac{x - x_{j-1}(w_{\gamma_k})}{x_j(w_{\gamma_k}) - x_{j-1}(w_{\gamma_k})} \frac{\partial}{\partial w} x_j(w_{\gamma_k})(w - w_{\gamma_k}) + \frac{\partial}{\partial x}(Y_j(\bar{t}, x; w_{\gamma_k}) - \bar{y}(x))$$

$$\left. \cdot \frac{x_j(w_{\gamma_k}) - x}{x_j(w_{\gamma_k}) - x_{j-1}(w_{\gamma_k})} \frac{\partial}{\partial w} x_{j-1}(w_{\gamma_k}) \cdot (w - w_{\gamma_k}) \right] \lambda_j(x; w_{\gamma_k}) \, dx \leq C.$$

For $j = 1, \ldots, K + 1$ we define the sets $\hat{I}_j \subset [x_{j-1}(\bar{w}) - \varepsilon_0, x_j(\bar{w}) + \varepsilon_0]$ by

$$(73) \qquad \hat{I}_j := \{x \in [x_{j-1}(\bar{w}) - \varepsilon_0, x_j(\bar{w}) + \varepsilon_0] : Y_j(\bar{t}, x, \bar{w}) \geq \bar{y}(x)\},$$

where $\varepsilon_0 > 0$ is chosen small enough such that the extensions $Y_j(\bar{t}, \cdot, \bar{w})$ are well defined on $[x_{j-1}(\bar{w}) - \varepsilon_0, x_j(\bar{w}) + \varepsilon_0]$. Furthermore, we define

$$\hat{I}_{j,\varepsilon_1} = \bigcup_{x \in \hat{I}_j} (x - \varepsilon_1, x + \varepsilon_1) \cap [x_{j-1}(\bar{w}) - \varepsilon_0, x_j(\bar{w}) + \varepsilon_0], \quad j = 1, \ldots, K + 1$$

for some $\varepsilon_1 > 0$. Since the union of arbitrary many open sets is open, the sets

$$[x_{j-1}(\bar{w}) - \varepsilon_0, x_j(\bar{w}) + \varepsilon_0] \setminus \hat{I}_{j,\varepsilon_1} = [x_{j-1}(\bar{w}) - \varepsilon_0, x_j(\bar{w}) + \varepsilon_0] \setminus \bigcup_{x \in \hat{I}_j} (x - \varepsilon_1, x + \varepsilon_1)$$

are closed and bounded and hence compact. Therefore and since the functions $(Y_j(\bar{t}, \cdot, \bar{w}) - \bar{y}(\cdot))$ are continuous, (73) yields that there exists $\delta_0 > 0$ with

$$Y_j(\bar{t}, x, \bar{w}) - \bar{y}(x) \leq -\delta_0 < 0 \quad \text{for all } x \in [x_{j-1}(\bar{w}) - \varepsilon_0, x_j(\bar{w}) + \varepsilon_0] \setminus \hat{I}_{j,\varepsilon_1}.$$

The continuity of $Y_j(\bar{t}, x; w)$ and $x_j(w)$ w.r.t. w and $w_{\gamma_k} \to \bar{w}$ yield that

$$(74) \qquad Y_j(\bar{t}, x; w_{\gamma_k}) - \bar{y}(x) \leq -\frac{\delta_0}{2} < 0 \ \text{ for all } \ x \in [x_{j-1}(w_{\gamma_k}), x_j(w_{\gamma_k})] \setminus \hat{I}_{j,\varepsilon_1}$$

holds true if k is large enough. Therefore, (74) and (67) yield that

$$(75) \qquad \lambda_j(x, \lambda_{\gamma_k}) = 0 \quad \text{for all } x \in \mathbb{R} \setminus \hat{I}_{j,\varepsilon_1}, \ j = 1, \ldots, K + 1$$

holds true for sufficiently large k. Our goal is to show that the sequences $(\lambda_j(\cdot; w_{\gamma_k}))_{k \in \mathbb{N}}$ are uniformly bounded in $L^1([a, b])$ for all $j = 1, \ldots, K + 1$. Since (23) is satisfied in $\bar{w}$ by assumption, Lemma 28 yields the existence of $\tilde{w} \in W_{ad}$ and $\varepsilon_2 > 0$ such that

$$(76) \quad Y_j(\bar{t}, x, \bar{w}) - \bar{y}(x) + \frac{\partial}{\partial w} Y_j(\bar{t}, x, \bar{w})(\tilde{w} - \bar{w}) + \frac{\partial}{\partial x}[Y_j(\bar{t}, x, \bar{w}) - \bar{y}(x)]\Big($$

$$\frac{x - x_{j-1}(\bar{w})}{x_j(\bar{w}) - x_{j-1}(\bar{w})} \frac{\partial}{\partial w} x_j(\bar{w})(\tilde{w} - \bar{w}) + \frac{x_j(\bar{w}) - x}{x_j(\bar{w}) - x_{j-1}(\bar{w})} \frac{\partial}{\partial w} x_{j-1}(\bar{w})(\tilde{w} - \bar{w})\Big) \leq -\varepsilon_2$$

holds true for all $x \in [x_{j-1}(\bar{w}), x_j(\bar{w})]$ and $j = 1, \ldots, K + 1$. Using the continuity of the terms in (76) w.r.t. x, there exists a constant ε_3 with $0 < \varepsilon_3 < \varepsilon_0$ such that (76) holds true for all $x \in [x_{j-1}(\bar{w}) - \varepsilon_3, x_j(\bar{w}) + \varepsilon_3]$ with $\frac{\varepsilon_2}{2}$ instead of ε_2 on the right side. Since it holds true that $(Y_j(\bar{t}, x, \bar{w})) - \bar{y}(x)\big|_{\hat{I}_j \cap [x_{j-1}(\bar{w}) - \varepsilon_3, x_j(\bar{w}) + \varepsilon_3]} \geq 0$, defining

$$(77) \qquad R_j(x, w) := \frac{\partial}{\partial w} Y_j(\bar{t}, x; w)(\tilde{w} - w) + \frac{\partial}{\partial x}[Y_j(\bar{t}, x; w) - \bar{y}(x)]\Big($$

$$\frac{x - x_{j-1}(w)}{x_j(w) - x_{j-1}(w)} \frac{\partial}{\partial w} x_j(w)(\tilde{w} - w) + \frac{x_j(w) - x}{x_j(w) - x_{j-1}(w)} \frac{\partial}{\partial w} x_{j-1}(w)(\tilde{w} - w)\Big)$$

for $j = 1, \ldots, K + 1$, we can further deduce that

$$(78) \qquad R_j(x, \bar{w}) \leq -\frac{\varepsilon_2}{2} \quad \forall x \in \hat{I}_j \cap [x_{j-1}(\bar{w}) - \varepsilon_3, x_j(\bar{w}) + \varepsilon_3].$$

By continuity of $R_j(\cdot, \bar{w})$ w.r.t. x, we obtain for sufficiently small ε_1 the estimation

$$(79) \qquad R_j(x, \bar{w}) \leq -\frac{\varepsilon_2}{4} \quad \forall x \in \hat{I}_{j,\varepsilon_1} \cap [x_{j-1}(\bar{w}) - \varepsilon_3, x_j(\bar{w}) + \varepsilon_3].$$

The continuity of $R_j(x, \cdot)$ and $x_j(\cdot)$ w.r.t. w and $w_{\gamma_k} \to \bar{w}$ yield for sufficiently large k

$$(80) \qquad R_j(x, w_{\gamma_k}) \leq -\frac{\varepsilon_2}{8} \quad \forall x \in \hat{I}_{j,\varepsilon_1} \cap [x_{j-1}(w_{\gamma_k}) - \frac{\varepsilon_3}{2}, x_j(w_{\gamma_k}) + \frac{\varepsilon_3}{2}].$$

From (75), we deduce for sufficiently large k and $j = 1, \ldots, K+1$

$$\int_{x_{j-1}(w_{\gamma_k})}^{x_j(w_{\gamma_k})} -R_j(x; w_{\gamma_k}) \lambda_j(x; w_{\gamma_k}) \, dx = \int_{[x_{j-1}(w_{\gamma_k}), x_j(w_{\gamma_k})] \cap \hat{I}_{j,\varepsilon_1}} -R_j(x; w_{\gamma_k}) \lambda_j(x; w_{\gamma_k}) \, dx.$$

Using (80) and the nonnegativity of $\lambda_j(x; w_{\gamma_k})$, we further obtain

$$\int_{[x_{j-1}(w_{\gamma_k}), x_j(w_{\gamma_k})] \cap \hat{I}_{j,\varepsilon_1}} -R_j(x; w_{\gamma_k}) \lambda_j(x; w_{\gamma_k}) \, dx \geq \int_{[x_{j-1}(w_{\gamma_k}), x_j(w_{\gamma_k})] \cap \hat{I}_{j,\varepsilon_1}} \frac{\varepsilon_2}{8} \lambda_j(x; w_{\gamma_k}) \, dx.$$

Using (75) again, we can conclude that

$$\int_{[x_{j-1}(w_{\gamma_k}), x_j(w_{\gamma_k})] \cap \hat{I}_{j,\varepsilon_1}} \frac{\varepsilon_2}{8} \lambda_j(x; w_{\gamma_k}) \, dx = \int_{x_{j-1}(w_{\gamma_k})}^{x_j(w_{\gamma_k})} \frac{\varepsilon_2}{8} \lambda_j(x; w_{\gamma_k}) \, dx = \int_{[a,b]} \frac{\varepsilon_2}{8} \lambda_j(x; w_{\gamma_k}) \, dx.$$

Hence, there exists $\bar{k} \in \mathbb{N}$ large enough such that for $j = 1, \ldots, K+1$ we obtain

$$(81) \qquad \int_{x_{j-1}(w_{\gamma_k})}^{x_j(w_{\gamma_k})} -R_j(x; w_{\gamma_k}) \lambda_j(x; w_{\gamma_k}) dx \geq \int_{[a,b]} \frac{\varepsilon_2}{8} \lambda_j(x; w_{\gamma_k}) \, dx \geq 0 \quad \forall k \geq \bar{k},$$

where the last inequality holds due to $\lambda_j(x; w_{\gamma_k}) \geq 0$. Since the left side of (81) is equal to the jth summand of the left side of (72), (72) and (81) yield

$$\sum_{j=1}^{K+1} \int_{[a,b]} |\lambda_j(x; w_{\gamma_k})| \, dx \leq \frac{8C}{\varepsilon_2} := \tilde{C} \quad \forall k \geq \bar{k}.$$

Thus, the sequences $(\lambda_j(x; w_{\gamma_k}))_{k \in \mathbb{N}}$ are uniformly bounded in $L^1([a,b])$. □

THEOREM 41. *Let (A1) and (A2) hold and consider a sequence $(w_{\gamma_k})_{k \in \mathbb{N}}$ of local solutions for (P_{γ_k}) converging to a local solution $\bar{w} \in W_{ad}$ for (P). If $\bar{w}$ satisfies (ND) and (23), then there exists a subsequence (again denoted by $(w_{\gamma_k})_{k \in \mathbb{N}}$) such that $\lambda_j(\cdot; w_{\gamma_k}) \to \mu_j$ in $\mathcal{M}([a,b])$-weak* for $j = 1, \ldots, K+1$. Hereby, the measures $\mu_j \in \mathcal{M}([a,b])$ are nonnegative and the optimality conditions in Theorem 24 are satisfied in $\bar{w}$ for $\bar{\mu}_j = \mu_j|_{I_j}$ for all $j = 1, \ldots, K+1$.*

Proof. Since the sequences of Lagrange multiplier estimates $(\lambda_j(\cdot; w_{\gamma_k}))_{k \in \mathbb{N}}$ are uniformly bounded in $L^1([a,b])$ by Lemma 40, the Banach-Alaoglu Theorem yields that there exists subsequences, again denoted by $(\lambda_j(\cdot; w_{\gamma_k}))_{k \in \mathbb{N}}$, such that $\lambda_j(\cdot; w_{\gamma_k}) \to \mu_j$ in $\mathcal{M}([a,b])$-weak* for $j = 1, \ldots, K+1$. Hereby, the measures $\mu_j \in \mathcal{M}([a,b])$ are nonnegative due to the fact that $(\lambda_j(\cdot; w_{\gamma_k})) \geq 0$ for all $k \in \mathbb{N}$. We will prove that the optimality conditions in Theorem 24, i.e. (35), (36) and (37) are satisfied if we choose $\bar{\mu}_j = \mu_j|_{I_j}$ for $j = 1, \ldots, K+1$. We note that since $\bar{w}$ is a local solution for (P), (35) holds true. Concerning (36), using that $0 \leq J_\gamma(w_{\gamma_k}) \leq J_\gamma(\bar{w}) = J(y(\bar{w}))$, $w_{\gamma_k} \to \bar{w}$ w.r.t. $\| \cdot \|_W$ and the definitions of $\lambda_j(\cdot; w)$ and $J_{\gamma_k}(y(w))$, we obtain that

$$0 \leq \lim_{k \to \infty} \sum_{j=1}^{K+1} \int_{x_{j-1}(\bar{w})}^{x_j(\bar{w})} (Y_j(\bar{t}, x; w_{\gamma_k}) - \bar{y}(x)) \lambda_j(x; w_{\gamma_k}) \, dx \leq 2 \cdot \lim_{k \to \infty} (J(\bar{w}) - J(w_{\gamma_k})) = 0.$$

Using this result, we further deduce that

$$0 = \lim_{k\to\infty} \sum_{j=1}^{K+1} \int_{x_{j-1}(\bar{w})}^{x_j(\bar{w})} (Y_j(\bar{t},x;w_{\gamma_k}) - \bar{y}(x)) \cdot \lambda_j(x;w_{\gamma_k}) \, \mathrm{d}x$$

$$= \lim_{k\to\infty} \sum_{j=1}^{K+1} \int_{x_{j-1}(\bar{w})}^{x_j(\bar{w})} (Y_j(\bar{t},x,\bar{w}) - \bar{y}(x)) \cdot \lambda_j(x;w_{\gamma_k}) \, \mathrm{d}x$$

$$(82) \qquad + \lim_{k\to\infty} \sum_{j=1}^{K+1} \int_{x_{j-1}(\bar{w})}^{x_j(\bar{w})} (Y_j(\bar{t},x;w_{\gamma_k}) - Y_j(\bar{t},x,\bar{w})) \cdot \lambda_j(x;w_{\gamma_k}) \, \mathrm{d}x.$$

We observe that the sequences $(\lambda_j(\cdot;w_{\gamma_k}))_{k\in\mathbb{N}}$ are uniformly bounded in $L^1([a,b])$ and the mappings $B_\varepsilon(\bar{w}) \ni w \mapsto Y_j(\bar{t},\cdot;w) \in C([x_{j-1}(\bar{w}), x_j(\bar{w})])$ are continuous for sufficiently small $\varepsilon > 0$. Hence, due to $w_{\gamma_k} \to \bar{w}$ it holds true that

$$(83) \qquad \lim_{k\to\infty} \sum_{j=1}^{K+1} \int_{x_{j-1}(\bar{w})}^{x_j(\bar{w})} (Y_j(\bar{t},x;w_{\gamma_k}) - Y_j(\bar{t},x,\bar{w})) \cdot \lambda_j(x;w_{\gamma_k}) \, \mathrm{d}x = 0.$$

Inserting (83) in (82) and using the fact that $\lambda_j(\cdot;w_{\gamma_k}) \to \mu_j$ in $\mathcal{M}([a,b])$-weak* for all $j = 1, \ldots, K+1$, we can conclude that

$$0 = \lim_{k\to\infty} \sum_{j=1}^{K+1} \int_{x_{j-1}(\bar{w})}^{x_j(\bar{w})} (\bar{y}(x) - Y_j(\bar{t},x,\bar{w})) \cdot \lambda_j(x;w_{\gamma_k}) \, \mathrm{d}x$$

$$(84) \qquad = \sum_{j=1}^{K+1} \int_{x_{j-1}(\bar{w})}^{x_j(\bar{w})} (\bar{y}(x) - Y_j(\bar{t},x,\bar{w})) \, \mathrm{d}\mu_j(x)$$

and hence (36) is proved. To show that (37) holds true, we use the abbreviations:

$$\bar{I}_1^j(x;w) := \frac{\partial}{\partial x} (\bar{y}(x) - Y_j(\bar{t},x;w)) \cdot \frac{x - x_{j-1}(w)}{x_j(w) - x_{j-1}(w)}$$

$$\bar{I}_2^j(x;w) := \frac{\partial}{\partial x} (\bar{y}(x) - Y_j(\bar{t},x;w)) \cdot \frac{x_j(w) - x}{x_j(w) - x_{j-1}(w)}$$

We note that the optimal solutions w_{γ_k} for (P_{γ_k}) satisfy (ND) for sufficiently large k due to Lemma 36. Therefore, using Theorem 38 and Lemma 39, we obtain that

$$\lim_{k\to\infty} \left[\frac{\partial}{\partial w} J(y(w_{\gamma_k}))(w - w_{\gamma_k}) + \sum_{j=1}^{K+1} \left[\int_{x_{j-1}(w_{\gamma_k})}^{x_j(w_{\gamma_k})} \frac{\partial}{\partial w} Y_j(\bar{t},x;w_{\gamma_k})(w - w_{\gamma_k})\lambda_j(x;w_{\gamma_k}) \, \mathrm{d}x \right. \right.$$

$$+ \int_{x_{j-1}(w_{\gamma_k})}^{x_j(w_{\gamma_k})} \bar{I}_1^j(w_{\gamma_k},x) \cdot \lambda_j(x;w_{\gamma_k}) \, \mathrm{d}x \cdot \frac{\partial}{\partial w} x_j(w_{\gamma_k})(w - w_{\gamma_k})$$

$$+ \int_{x_{j-1}(w_{\gamma_k})}^{x_j(w_{\gamma_k})} \bar{I}_2^j(w_{\gamma_k},x) \cdot \lambda_j(x;w_{\gamma_k}) \, \mathrm{d}x \cdot \frac{\partial}{\partial w} x_{j-1}(w_{\gamma_k})(w - w_{\gamma_k}) \right] \right] \geq 0 \ \forall w \in W_{ad}.$$

In order to replace the integration limits $x_j(w_{\gamma_k})$ by $x_j(\bar{w})$ for all $j = 1, \ldots, K+1$, we use the variable transformation

$$(85) \qquad x = x_{j-1}(w_{\gamma_k}) + \frac{\tilde{x} - x_{j-1}(\bar{w})}{x_j(\bar{w}) - x_{j-1}(\bar{w})} \left(x_j(w_{\gamma_k}) - x_{j-1}(w_{\gamma_k}) \right)$$

and obtain

$$\lim_{k\to\infty}\Big[\frac{\partial}{\partial w}J(y(w_{\gamma_k}))(w-w_{\gamma_k})+\sum_{j=1}^{K+1}\Big[\int_{x_{j-1}(\bar w)}^{x_j(\bar w)}\tilde Y_j(\bar t,\tilde x;w_{\gamma_k})(w-w_{\gamma_k})$$

$$\frac{x_j(w_{\gamma_k})-x_{j-1}(w_{\gamma_k})}{x_j(\bar w)-x_{j-1}(\bar w)}\tilde\lambda_j(\tilde x;w_{\gamma_k})\,\mathrm{d}\tilde x+\int_{x_{j-1}(\bar w)}^{x_j(\bar w)}\tilde I_1^j(\tilde x;w_{\gamma_k})\cdot\frac{x_j(w_{\gamma_k})-x_{j-1}(w_{\gamma_k})}{x_j(\bar w)-x_{j-1}(\bar w)}$$

$$\tilde\lambda_j(\tilde x;w_{\gamma_k})\,\mathrm{d}\tilde x\cdot\frac{\partial}{\partial w}x_j(w_{\gamma_k})(w-w_{\gamma_k})+\int_{x_{j-1}(\bar w)}^{x_j(\bar w)}\tilde I_2^j(\tilde x;w_{\gamma_k})$$

$$(86)\quad\cdot\frac{x_j(w_{\gamma_k})-x_{j-1}(w_{\gamma_k})}{x_j(\bar w)-x_{j-1}(\bar w)}\tilde\lambda_j(\tilde x;w_{\gamma_k})\,\mathrm{d}\tilde x\cdot\frac{\partial}{\partial w}x_{j-1}(w_{\gamma_k})(w-w_{\gamma_k})]\Big]\geq0,$$

with $\;\dot:=x_{j-1}(w_{\gamma_k})+\dfrac{\tilde x-x_{j-1}(\bar w)}{x_j(\bar w)-x_{j-1}(\bar w)}\big(x_j(w_{\gamma_k})-x_{j-1}(w_{\gamma_k})\big)\;$ and

$$(87)\quad \tilde Y_j(\bar t,\tilde x;w_{\gamma_k}):=\frac{\partial}{\partial w}Y_j(\bar t,\dot:;w_{\gamma_k}),\;\tilde I_i^j(\tilde x;w_{\gamma_k}):=\bar I_i^j(\dot:;w_{\gamma_k}),\;\tilde\lambda_j(\tilde x;w_{\gamma_k}):=\lambda_j(\dot:;w_{\gamma_k})$$

for $j=1,\ldots,K+1$ and $i=1,2$. Since the sequences $\big(\lambda_j(x;w_{\gamma_k})\big)_{k\in\mathbb N}$ are bounded in $L^1([a,b])$ by Lemma 40, the sequences $\big(\tilde\lambda_j(\cdot;w_{\gamma_k})\big)_{k\in\mathbb N}$ are also bounded in $L^1([a,b])$. Due to this result, by the Banach-Alaoglu Theorem one can deduce that there exists another subsequence (again denoted by $\big(\tilde\lambda_j(\cdot;w_{\gamma_k})\big)_{k\in\mathbb N}$) with

$$(88)\qquad\tilde\lambda_j(\cdot;w_{\gamma_k})\to\tilde\mu_j\text{ in }\mathcal M([a,b])\text{-weak}^*\quad\text{for all }j=1,\ldots,K+1,$$

where $\tilde\mu_j\in\mathcal M([a,b])$ are nonnegative. From Lemma 36, we further obtain that

$$(89)\qquad\tilde Y_j(\bar t,\cdot;w_{\gamma_k})\to\frac{\partial}{\partial w}Y_j(\bar t,\cdot;\bar w),\;\tilde I_i^j(\cdot;w_{\gamma_k})\to\bar I_i^j(\cdot;\bar w)\text{ in }C([x_{j-1}(\bar w),x_j(\bar w)])$$

for $k\to\infty$. Using (88), (89) and $\frac{x_j(w_{\gamma_k})-x_{j-1}(w_{\gamma_k})}{x_j(\bar w)-x_{j-1}(\bar w)}\to1$, (86) can be rewritten as

$$J(y(\bar w))(w-\bar w)+\sum_{j=1}^{K+1}\Big[\int_{x_{j-1}(\bar w)}^{x_j(\bar w)}\frac{\partial}{\partial w}Y_j(\bar t,x,\bar w)(w-\bar w)\,\mathrm{d}\tilde\mu_j(x)+\int_{x_{j-1}(\bar w)}^{x_j(\bar w)}\bar I_1^j(\bar w,x)\,\mathrm{d}\tilde\mu_j(x)$$

$$\cdot\frac{\partial}{\partial w}x_j(\bar w)(w-\bar w)+\int_{x_{j-1}(\bar w)}^{x_j(\bar w)}\bar I_2^j(\bar w,x)\,\mathrm{d}\tilde\mu_j(x)\cdot\frac{\partial}{\partial w}x_{j-1}(\bar w)(w-\bar w)\Big]\geq0.$$

Hence (37) is satisfied if we choose $\bar\mu_j=\tilde\mu_j|_{I_j}$ for all $j=1,\ldots,K+1$. We finally show that (37) is also satisfied for the choice $\bar\mu_j=\mu_j|_{I_j}$. To this end, we prove that $\tilde\mu_j=\mu_j$ holds true in $\mathcal M(I_j)$ for $j=1,\ldots,K+1$. We consider compact intervals $J_{j,\varepsilon}=[x_{j-1}(\bar w)-\varepsilon,x_j(\bar w)+\varepsilon]\supset I_j$ for some small $\varepsilon>0$. For sufficiently large k it holds true that

$$x_{j-1}(\bar w)-\varepsilon\leq x_{j-1}(w_{\gamma_k})<x_j(w_{\gamma_k})\leq x_j(\bar w)+\varepsilon\quad\text{for all }j=1,\ldots,K+1.$$

We note that the weak-* convergence of $(\lambda_j(\cdot;w_{\gamma_k}))_{k\in\mathbb N}$ to $\mu_j\in\mathcal M([a,b])$ implies that for all $j=1,\ldots,K+1$ and for arbitrary $\varphi_j\in C(J_{j,\varepsilon})$ it holds true that

$$(90)\qquad\int_{J_{j,\varepsilon}}\varphi_j(x)\,\mathrm{d}\mu_j(x)=\lim_{k\to\infty}\int_{J_{j,\varepsilon}}\varphi_j(x)\lambda_j(x;w_{\gamma_k})\,\mathrm{d}x.$$

The fact that $(x_{j-1}(w_{\gamma_k}), x_j(w_{\gamma_k})) \subset J_{j,\varepsilon}$ for $j = 1, \ldots, K+1$ and (67) yield

$$(91) \quad \int_{J_{j,\varepsilon}} \varphi_j(x)\,\mathrm{d}\mu_j(x) = \lim_{k\to\infty} \int_{J_{j,\varepsilon}} \varphi_j(x)\lambda_j(x; w_{\gamma_k})\,\mathrm{d}x = \lim_{k\to\infty} \int_{x_{j-1}(w_{\gamma_k})}^{x_j(w_{\gamma_k})} \varphi_j(x)\lambda_j(x; w_{\gamma_k})\,\mathrm{d}x.$$

Using again the variable transformation in (85), we further obtain from (91)

$$(92) \quad \int_{J_{j,\varepsilon}} \varphi_j(x)\,\mathrm{d}\mu_j(x) = \lim_{k\to\infty} \int_{x_{j-1}(\bar{w})}^{x_j(\bar{w})} \tilde{\varphi}_j(\tilde{x})\tilde{\lambda}_j(\tilde{x}; w_{\gamma_k})\,\mathrm{d}\tilde{x}, \quad \text{where}$$

$$\tilde{\varphi}_j(\tilde{x}) = \varphi_j\left(x_{j-1}(w_{\gamma_k}) + \frac{\tilde{x} - x_{j-1}(\bar{w})}{x_j(\bar{w}) - x_{j-1}(\bar{w})}\left(x_j(w_{\gamma_k}) - x_{j-1}(w_{\gamma_k})\right)\right) \cdot \frac{x_j(w_{\gamma_k}) - x_{j-1}(w_{\gamma_k})}{x_j(\bar{w}) - x_{j-1}(\bar{w})}$$

and $\tilde{\lambda}$ is defined as in (87). Hereby, the definition of $\tilde{\lambda}$ in (87) yields that

$$(93) \quad \int_{J_{j,\varepsilon}} \varphi_j(x)\,\mathrm{d}\mu_j(x) = \lim_{k\to\infty} \int_{x_{j-1}(\bar{w})}^{x_j(\bar{w})} \tilde{\varphi}_j(\tilde{x})\tilde{\lambda}_j(\tilde{x}; w_{\gamma_k})\,\mathrm{d}\tilde{x} = \lim_{k\to\infty} \int_{J_{j,\varepsilon}} \tilde{\varphi}_j(\tilde{x})\tilde{\lambda}_j(\tilde{x}; w_{\gamma_k})\,\mathrm{d}\tilde{x}.$$

Due to $\varphi_j \in C(J_{j,\varepsilon})$ and the fact that $x_j(w_{\gamma_k}) \to x_j(\bar{w})$ for $k \to \infty$, we obtain that $\tilde{\varphi}_j(\tilde{x})$ converge to $\varphi_j(\cdot)$ in $C(J_{j,\varepsilon})$. Using this result, the uniform boundedness of $(\tilde{\lambda}_j(\cdot; w_{\gamma_k}))_{k\in\mathbb{N}}$ in $L^1([a,b])$ and $\tilde{\lambda}_j(\cdot; w_{\gamma_k}) \to \tilde{\mu}_j$ in $\mathcal{M}([a,b])$-weak*, where $\tilde{\mu}_j \in \mathcal{M}([a,b])$ are nonnegative, we obtain from (93) for all $j = 1, \ldots, K+1$ that

$$\int_{J_{j,\varepsilon}} \varphi_j(x)\,\mathrm{d}\mu_j(x) = \lim_{k\to\infty} \int_{J_{j,\varepsilon}} \tilde{\varphi}_j(\tilde{x})\tilde{\lambda}_j(\tilde{x}; w_{\gamma_k})\mathrm{d}\tilde{x} = \int_{J_{j,\varepsilon}} \varphi_j(\tilde{x})\tilde{\mu}_j(\tilde{x})\mathrm{d}\tilde{x}$$

holds true. Since $\varphi_j \in C(J_{j,\varepsilon})$ were arbitrary chosen it holds that $\tilde{\mu}_j|_{I_j} = \mu_j|_{I_j}$ in $\mathcal{M}(J_{j,\varepsilon})$ and hence (37) is also satisfied if we choose $\bar{\mu}_j = \mu_j|_{I_j}$ for $j = 1, \ldots, K+1$.□

5. Conclusion and possible extensions. We have considered the optimal control of initial-boundary value problems with pointwise state constraints. Hereby, we have proved the existence of an optimal control and have derived necessary optimality conditions. In addition, we have discussed the Moreau-Yosida regularization approach where we have proved that each sequence of optimal solutions for the regularized problems converges strongly to an optimal solution of the original problem with pointwise state constraints. Furthermore, we have derived optimality conditions for the regularized problems and have shown that if the optimal solutions of the regularized problems converge to a local optimum for (P) in which Robinson's CQ is satisfied, then there exists a sequence of Lagrange multiplier estimates that converges in $\mathcal{M}([a,b])$-weak* to Lagrange multipliers for the optimality system for (P). The method, that was developed in this paper to derive necessary optimality conditions, could be extended to networks with node conditions, e.g. traffic light problems, where the traffic flow is modelled by the LWR-modell (see [20, 26]) by using the results of [25]. Another possible extension is to consider systems of balance laws, e.g. the Euler equations modelling the gas flow in a pipe. Hereby, if the solultion has similar structural properties, that we have collected in section 2 (see also condition H2 in [5]), than one can introduce new state variables and derive optimality conditions.

Appendix A. Proof of Theorem 13.

Proof. In the following we will prove Fréchet-differentiability of the mapping (11) in $w = \bar{w}$, where the continuous differentiability follows from the stability of genuine characteristics and of the adjoint state (cf. [23, Proof of Lemma 6.2.7]). Denote by $\xi_{l/r}$ the minimal/maximal backward characteristic through $(\bar{t}, x_s(\bar{w}))$. We will restrict ourselves in the proof to the case that ξ_l ends in the interior of a rarefaction wave created by a discontinuity of the boundary data $u_{B,\mathfrak{a}}$ in $t = \bar{t}_m^{\mathfrak{a}}$ and ξ_r ends in a point $(0, z)$ where the initial data u_0 is smooth. We further assume that $(\bar{t}_m^{\mathfrak{a}}, \mathfrak{a})$ and $(0, \bar{x}_l^0)$ with $\bar{x}_l^0 < z$ are the only rarefaction wave creating discontinuities of the initial and boundary data. Moreover, we assume that

$$(94) \qquad \mathbb{T}^{\mathfrak{a}} = \emptyset,$$

where the treatment of the case that $\mathbb{T}^{\mathfrak{a}} \neq \emptyset$ is explained in [24, Lemma 4.13]. We note that (6) and (94) together yield

$$(95) \qquad y(\cdot, 0+; w) = u_m^{B,\mathfrak{a}}(\cdot) \quad \text{on } [0, \bar{t}]$$

for all $w \in B_\rho^W(\bar{w})$ with $\rho > 0$ small enough. We use these assumptions for the sake of simplicity and in order to avoid technical effort. Nevertheless, the proof can easily be extended to the general case as we will show later.

Consider some $\delta w \in W$ and set $w := \bar{w} + \delta w$. Furthermore, let $\bar{y} := y(\bar{w})$ and $y := y(w)$ denote the entropy solutions of (1) for the controls $\bar{u}_1$ and u_1 in the source term and initial and boundary values $\bar{u}_0 := u_0(\bar{w})$, $\bar{u}_{B,\mathfrak{a}/\mathfrak{b}} := u_{B,\mathfrak{a}/\mathfrak{b}}(\bar{w})$, $u_0 := u_0(w)$ and $u_{B,\mathfrak{a}/\mathfrak{b}} := u_{B,\mathfrak{a}/\mathfrak{b}}(w)$, respectively. We define $\delta u_0 := u_0 - \bar{u}_0$, $\delta u_{B,\mathfrak{a}/\mathfrak{b}} := u_{B,\mathfrak{a}/\mathfrak{b}} - \bar{u}_{B,\mathfrak{a}/\mathfrak{b}}$, $\delta u_1 = u_1 - \bar{u}_1$ and $\Delta y := y - \bar{y}$.

In the following, C and ρ denote large/small constants, respectively, which possibly change their values throughout the proof. For the sake of simplicity, we set $\mathfrak{a} = 0$.

As in [23, Proof of Lemma 6.3.7], one can show that for all $\bar{\varepsilon} > 0$ it holds true that

$$(96) \qquad \int_{x_s(\bar{w})-\bar{\varepsilon}}^{x_s(\bar{w})+\bar{\varepsilon}} \Delta y(\bar{t}, x)\,dx = (x_s(w) - x_s(\bar{w}))[\bar{y}(\bar{t}, x_s(\bar{w}))] + O((\bar{\varepsilon} + \|\delta w\|_W)\|\delta w\|_W).$$

In the remaining part of the proof, we will derive an adjoint representation for the term

$$(97) \qquad \frac{1}{[\bar{y}(\bar{t}, x_s(\bar{w}))]} \int_{x_s(\bar{w})-\bar{\varepsilon}}^{x_s(\bar{w})+\bar{\varepsilon}} \Delta y(\bar{t}, x)\,dx.$$

As in [24, Proof of Lemma 4.10], we define for $(t, x) \in \Omega_{\bar{t}} := (0, \bar{t}) \times (0, \mathfrak{b})$

$$a(t, x) := f'(\bar{y}(t, x)), \quad b(t, x) := g_y(t, x, \bar{y}(t, x), \bar{u}_1),$$

$$(98) \quad \tilde{a}(t, x) := \int_0^1 f'(\tau y(t, x) + (1 - \tau)\bar{y}(t, x))\,d\tau, \quad \tilde{b}(t, x) := g_y(t, x, y(t, x), u_1)$$

and observe that Δy is a weak solution of

$$(99) \qquad \Delta y_t + (\tilde{a}\Delta y)_x = \tilde{b}\Delta y + g(\cdot, \bar{y}, u_1) - g(\cdot, \bar{y}, \bar{u}_1)$$

on $\Omega_{\bar{t}}$. As described in [24, Lemma 4.10], the functions $a, \tilde{a}, b, \tilde{b}$ can be extended to $[0, \bar{t}] \times \mathbb{R}$ by setting

$$a(t, x) = \tilde{a}(t, x) = M_{f'} \text{ if } x < 0, \qquad (b, \tilde{b})(t, x) = (b, \tilde{b})(t, 0+) \text{ if } x < 0,$$

$$a(t, x) = \tilde{a}(t, x) = -M_{f'} \text{ if } x > \mathfrak{b} \qquad (b, \tilde{b})(t, x) = (b, \tilde{b})(t, \mathfrak{b}-) \text{ if } x > \mathfrak{b}.$$

We observe that since g_y does not depend on y by (A1) and due to the regularity of g, it holds true that

$$\tilde{b}, b \in L^\infty(0, \bar{t}; C^{0,1}(\mathbb{R})) \quad \text{and}$$

(100) $$\tilde{b} \to b \text{ in } L^\infty(0, \bar{t}; C(\mathbb{R})) \text{ as } \|\delta w\|_W \to 0$$

(cf. [29]).

Let the interval $[t_1, t_2] \subset [0, T]$ be chosen such that $\bar{y}$ has no rarefaction wave creating discontinuity on $[t_1, t_2] \times [0, \mathfrak{b}]$. Then Proposition 2 and Corollary 5 yield

$$\|\tilde{a}\|_{\infty, [t_1, t_2] \times \mathbb{R}}, \|a\|_{\infty, [t_1, t_2] \times \mathbb{R}} \leq M_y,$$

(101) $$\tilde{a} \to a \text{ in } L^1_{\text{loc}}([t_1, t_2] \times \mathbb{R}) \text{ and in } L^\infty([t_1, t_2] \times \mathbb{R}))\text{-weak}^* \text{as } \|\delta w\|_W \to 0.$$

(cf. [29]). Moreover, from (7) and [23, Lemma 6.3.3] we can deduce that there exists a constant $\rho > 0$ such that for all $w \in B^W_\rho(\bar{w})$ the coefficients $\tilde{a}(\cdot)$ and $a(\cdot)$ satisfy on $[t_1, t_2] \times \mathbb{R}$ the one sided Lipschitz condition (OSLC) (cf. [2]), i.e. there exists $\alpha \in L^1(t_1, t_2)$ such that

(102) $$a_x(t, \cdot) \leq \alpha(t) \text{ for a.a. } t \in [t_1, t_2]$$

holds true in the sense of distributions on $\mathbb{R}$.

We now consider

(103) $$p_t + \tilde{a}p_x = -\tilde{b}p \quad \text{on } (t_1, t_2) \times \mathbb{R}, \qquad p(t_2, \cdot) = p^{t_2}(\cdot) \quad \text{on } \mathbb{R}$$

(104) $$\text{and} \quad p_t + ap_x = -bp \quad \text{on } (t_1, t_2) \times \mathbb{R}, \qquad p(t_2, \cdot) = p^{t_2}(\cdot) \quad \text{on } \mathbb{R}.$$

Since (100), (101) and (102) hold, we can apply [29, Theorem 14] yielding that (103) and (104) admit reversible solutions

(105) $$\hat{p} \in C^{0,1}([t_1, t_2] \times \mathbb{R})$$

(106) $$p \in C^{0,1}([t_1, t_2] \times \mathbb{R}),$$

respectively. In what follows, a reversible solution has to be understood according to [29, Definition 12]. Moreover, (100), (101) and (102) yield

(107) $$\hat{p} \to p \text{ in } C([t_1, t_2] \times [-R, R])$$

for all $R > 0$ (cf. [29, Theorem 16]).

Before we start to derive an adjoint representation for the term in (97), we first give a brief overview of the main steps and basic concepts of the proof.

Since $\bar{w}$ satisfies (ND), for sufficiently small $\bar{\varepsilon}$ the points $x = x_s(\bar{w}) - \bar{\varepsilon}$ and $x = x_s(\bar{w}) + \bar{\varepsilon}$ are points of continuity of $\bar{y}(\bar{t}, \cdot)$. Therefore, there exist unique backward characteristics $\zeta_{l/r}$ through the points $x_s(\bar{w}) \mp \bar{\varepsilon}$ such that we can define the set

$$D^{\bar{\varepsilon}} := \{(t, x) \in [0, \bar{t}^{\mathfrak{a}}_m] \times [0, \mathfrak{b}] : 0 \leq x \leq \zeta_r(t)\}$$

(108) $$\cup \{(t, x) \in [\bar{t}^{\mathfrak{a}}_m, \bar{t}] \times [0, \mathfrak{b}] : \zeta_l(t) \leq x \leq \zeta_r(t)\}.$$

Due to the stability of backward characteristics it holds that

$$D^0 = \{(t, x) \in [0, \bar{t}^{\mathfrak{a}}_m] \times [0, \mathfrak{b}] : 0 \leq x \leq \xi_r(t)\}$$

(109) $$\cup \{(t, x) \in [\bar{t}^{\mathfrak{a}}_m, \bar{t}] \times [0, \mathfrak{b}] : \xi_l(t) \leq x \leq \xi_r(t)\}.$$

The main idea of this proof is equal to the proof of Lemma 4.10 in [24], but since in the present case the shifting of rarefaction centers in the initial and boundary data is allowed, we have to find a slightly different method to derive an adjoint representation of (97). Hereby, we will follow the ideas that are used in [25] and based on the fact that if the source term g is equal to zero, then the local solution near a rarefaction center is explicitly known. The proof consists of the following main steps (we highly recommend the reader to have a look at Figure 1):

In **step 1** we will choose some sufficiently small $\tilde{t} > \bar{t}_m^{\mathfrak{a}}$ with $\bar{t}_m^{\mathfrak{a}} < \tilde{t} < \bar{t}_{m+1}^0$ such that

$$(110) \qquad \tilde{x} := M_{f'}(\tilde{t} - \bar{t}_m^{\mathfrak{a}}) < \frac{\varepsilon_g}{2} \quad \text{and} \quad \bar{y}(\tilde{t}, \cdot) \in C^{0,1}([0, 2\tilde{x}])$$

holds and define the set

$$(111) \qquad D_{1,\bar{\varepsilon}} := \{(t, x) \in [\tilde{t}, \bar{t}] \times \Omega : \zeta_l(t) \leq x \leq \zeta_r(t)\}.$$

Then we consider (99) on $[\tilde{t}, \bar{t}] \times \Omega$, multiply it with the reversible solution $\hat{p}_1$ of (103) for $[t_1, t_2] = [\tilde{t}, \bar{t}]$ and enddata $p^{\bar{t}} = \frac{1}{[\bar{y}(\bar{t}, x_s(\bar{w}))]}$ and apply integration by parts on $D_{1,\bar{\varepsilon}}$. Since $[\tilde{t}, \bar{t}] \times \Omega$ contains no rarefaction center of $\bar{y}$, (102) holds such that using (107) yields that (97) is up to some $o(\|\delta w\|_W)$ and $\bar{\varepsilon} O(\|\delta w\|_W)$ terms equal to some terms $I_{1,\varepsilon} + I_{2,\varepsilon}$, which are defined in (127). The remaining steps are concerned with further simplifying $I_{1,\varepsilon}$ and $I_{2,\varepsilon}$.

In **step 2**, using the explicitly known local solution near the rarefaction center $\bar{t}_m^{\mathfrak{a}}$, we will derive a representation of $I_{1,\varepsilon}$ in (132), which depends on the local solution near the rarefaction center and the adjoint state in Definition 12.

Step 3 is concerned with the simplification of $I_{2,\varepsilon}$. To this end, we define the domains

$$(112) \quad D_{2,\varepsilon} := \{(t, x) \in [\hat{t}, \tilde{t}] \times [0, \mathfrak{b}] : \max\{0, f'(\bar{u}_m^{B,\mathfrak{a}}(\bar{t}_m^{\mathfrak{a}} - \varepsilon))(t - \bar{t}_m^{\mathfrak{a}} + \varepsilon)\} \leq x \leq \xi_r(t)\},$$

$$(113) \quad D_2 := \{(t, x) \in [\hat{t}, \tilde{t}] \times [0, \mathfrak{b}] : \max\{0, f'(\bar{u}_m^{B,\mathfrak{a}}(\bar{t}_m^{\mathfrak{a}}))(t - \bar{t}_m^{\mathfrak{a}})\} \leq x \leq \xi_r(t)\},$$

where $\hat{t} > \bar{t}_{m-1}^{\mathfrak{a}}$ and ε is chosen according to (128). We note that due to the rarefaction center of $\bar{y}$ in $(\bar{t}_m^{\mathfrak{a}}, 0)$ the OSLC (102) is violated. To solve this problem, we replace the coefficients a and $\tilde{a}$ on $[\hat{t}, \tilde{t}] \times \mathbb{R}$ by suitable $\tilde{a}_{loc}, a_{loc}$ such that the OSLC (102) is satisfied and in addition $\tilde{a}_{loc}|_{D_{2,\varepsilon}} \equiv \tilde{a}$ and $a_{loc}|_{D_{2,\varepsilon}} \equiv a$ hold for all $w \in B_\rho^W(\bar{w})$ with a sufficiently small $\rho > 0$. Let (99) be considered on $[\hat{t}, \tilde{t}] \times \Omega$, multiply it with the reversible solution $\hat{p}_2$ of (103) for $[t_1, t_2] = [\hat{t}, \tilde{t}]$, $\tilde{a}$ replaced by $\tilde{a}_{loc}$ and enddata $p^{\bar{t}} = p_1(\tilde{t}, \cdot)$. Then we apply integration by parts on $D_{2,\bar{\varepsilon}}$. Observing that (100), (101) and (102) are satisfied, we can use (107) to rewrite $I_{2,\varepsilon}$ in terms of the boundary data, the reversible solution p_2 of (104) with a replaced by a_{loc} and the term I_{21}, which is defined in (147).

In **step 4**, we have a closer look at the term I_{21}. Choosing a time point s with $0 < s < \varepsilon_g$, we define the set

$$(114) \qquad D_3 := \{(t, x) \in [s, \hat{t}] \times [0, \mathfrak{b}] : 0 \leq x \leq \xi_r(t)\}.$$

Since D_3 contains no rarefaction center, we can proceed as in step 1 and derive a representation for I_{21} (up to some $o(\|\delta w\|_W)$ terms) depending on the boundary data, the reversible solution p_3 of (104) for $[t_1, t_2] = [s, \hat{t}]$ and enddata $p^{\hat{t}} = p_2(\hat{t})$ and the terms $I_{1,\varepsilon}^0, I_{2,\varepsilon}^0$ and $I_{3,\varepsilon}^0$ which are defined in (163).

In **step 5**, we use the same methods as in step 2 to derive a representation of $I_{2,\varepsilon}^0$. Concerning $I_{1,\varepsilon}^0$ and $I_{3,\varepsilon}^0$, we can use similar arguments as in step 3.

Finally, setting $\bar{\varepsilon} := \bar{\varepsilon}(\delta w)$ as a function depending on δw and satisfying $\bar{\varepsilon}(\delta w) \to 0$ if $\|\delta w\|_W \to 0$, we obtain the desired result.
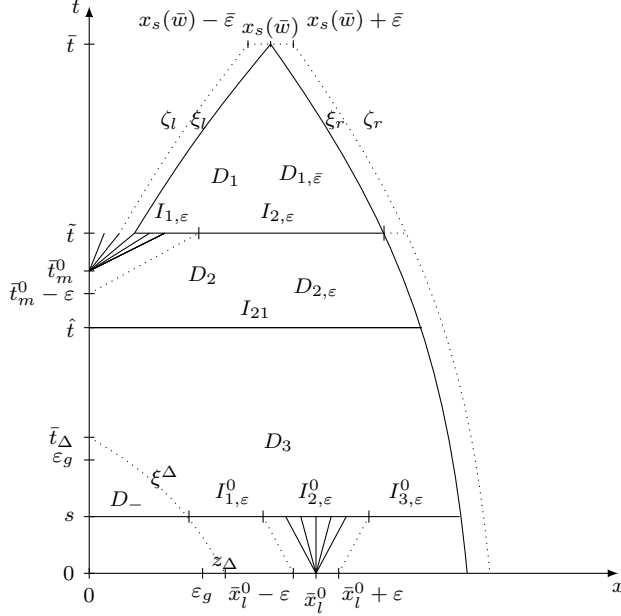

FIG. 1. *Proof of Theorem 13*

Step 1: We recall the set $D_{1,\bar{\varepsilon}}$ in (111) and observe that defining

$$D_1 := \{(t,x) \in [\tilde{t},\bar{t}] \times \Omega : \xi_l(t) \le x \le \xi_r(t)\},$$

it holds that $D_{1,0} = D_1$. Since $y(\cdot;w)$ has no rarefaction center on $D_{1,\bar{\varepsilon}}$ for all $w \in B_\rho^W(\bar{w})$ with $\rho > 0$ small enough, (100), (100) and (102) hold on $[\tilde{t},\bar{t}] \times \mathbb{R}$. Therefore,

$$(115) \qquad p_t + \tilde{a}p_x = -\tilde{b}p \quad \text{on } (\tilde{t},\bar{t}) \times \mathbb{R}, \quad p(\tilde{t},\cdot) = \frac{1}{[y(\bar{t},x_s(\bar{w});\bar{w})]} \quad \text{on } \mathbb{R}$$

$$(116) \quad \text{and } p_t + ap_x = -bp \quad \text{on } (\tilde{t},\bar{t}) \times \mathbb{R}, \quad p(\bar{t},\cdot) = \frac{1}{[y(\bar{t},x_s(\bar{w});\bar{w})]} \quad \text{on } \mathbb{R}$$

admit reservible solutions $\hat{p}_1, p_1 \in C^{0,1}([\tilde{t},\bar{t}] \times \mathbb{R})$ satisfying

$$(117) \qquad \hat{p}_1 \to p_1 \quad \text{in } C([\tilde{t},\bar{t}] \times [-R,R]) \quad \text{if } \|\delta w\|_W \to 0$$

for all $R > 0$. We consider (99) on $(\tilde{t},\bar{t}) \times \mathbb{R}$, multiply it with $\hat{p}_1$ and apply integration

by parts on the domain $D_{1,\bar\varepsilon}$ yielding

$$\frac{1}{[\bar{y}(\bar{t}, x_s(\bar{w}))]} \int_{x_s(\bar{w})-\bar\varepsilon}^{x_s(\bar{w})+\bar\varepsilon} \Delta y(\bar{t}, x)\,\mathrm{d}x = \int_{\zeta_l(\tilde{t})}^{\zeta_r(\tilde{t})} \hat{p}_1(\tilde{t}, x)\Delta y(\tilde{t}, x)\,\mathrm{d}x$$

$$+ \int_{D_{1,\bar\varepsilon}} \hat{p}_1(t, x)\big(g(t, x, \bar{y}, u_1) - g(t, x, \bar{y}, \bar{u}_1)\big)\,\mathrm{d}x\,\mathrm{d}t$$

$$- \int_{\tilde{t}}^{\bar{t}} \hat{p}_1(t, \zeta_l(t))\Delta y(t, \zeta_l(t))(\tilde{a}(t, \zeta_l(t)) - a(t, \zeta_l(t)))\,\mathrm{d}t$$

$$(118) \qquad + \int_{\tilde{t}}^{\bar{t}} \hat{p}_1(t, \zeta_r(t))\Delta y(t, \zeta_r(t))(\tilde{a}(t, \zeta_r(t)) - a(t, \zeta_r(t)))\,\mathrm{d}t.$$

Using (117) and the regularity of g w.r.t. u_1, we can rewrite the second term on the right hand side of (118) by

$$\int_{D_{1,\bar\varepsilon}} \hat{p}_1(t, x)\big(g(t, x, \bar{y}, u_1) - g(t, x, \bar{y}, \bar{u}_1)\big)\,\mathrm{d}x\,\mathrm{d}t$$

$$= \int_{D_{1,\bar\varepsilon}} p_1(t, x)g_{u_1}(t, x, \bar{y}, \bar{u}_1)\delta u_1\,\mathrm{d}x\,\mathrm{d}t + o(\|\delta w\|_W)$$

$$= \int_{D_{1,\bar\varepsilon}\setminus D_1} p_1(t, x)g_{u_1}(t, x, \bar{y}, \bar{u}_1)\delta u_1\,\mathrm{d}x\,\mathrm{d}t$$

$$+ \int_{D_1} p_1(t, x)g_{u_1}(t, x, \bar{y}, \bar{u}_1)\delta u_1\,\mathrm{d}x\,\mathrm{d}t + o(\|\delta w\|_W)$$

Hereby, the term $o(\|\delta w\|_W)$ is uniform w.r.t. $\bar\varepsilon > 0$. From [23, Lemma 3.1.15] we can deduce that

$$(119) \qquad\qquad \|\xi_{l/r}(\cdot) - \zeta_{l/r}(\cdot)\|_{C([\hat{t},\tilde{t}])} \le C\bar\varepsilon.$$

This and the boundedness of $g_{u_1}(\cdot, \bar{y}, \bar{u}_1)$ yield

$$\int_{D_{1,\bar\varepsilon}\setminus D_1} p_1(t, x)g_{u_1}(t, x, \bar{y}, \bar{u}_1)\delta u_1\,\mathrm{d}x\,\mathrm{d}t = \bar\varepsilon O(\|\delta w\|_W)$$

and hence

$$\int_{D_{1,\bar\varepsilon}} \hat{p}_1(t, x)\big(g(t, x, \bar{y}, u_1) - g(t, x, \bar{y}, \bar{u}_1)\big)\,\mathrm{d}x\,\mathrm{d}t = \int_{D_1} p_1(t, x)g_{u_1}(t, x, \bar{y}, \bar{u}_1)\delta u_1\,\mathrm{d}x\,\mathrm{d}t$$

$$(120) \qquad\qquad\qquad\qquad\qquad + \bar\varepsilon O(\|\delta w\|_W) + o(\|\delta w\|_W).$$

Since ξ_r ends in a point where $\bar{u}_0$ is smooth and ξ_l ends in the inner of a rarefaction center we can use [23, Lemma 6.2.1] yielding smooth local solutions $w \mapsto Y_{l/r}(\cdot, w)$ defined on some stripes $S_{l/r}$. Using those local solutions, the definitions of $\tilde{a}$ and a and the uniform boundedness of $\hat{p}_1$, we can show that the last two integrals on the right hand side of (118) are equal to $O(\|\delta w\|_W^2)$:

$$- \int_{\tilde{t}}^{\bar{t}} \hat{p}_1(t, \zeta_l(t))\Delta y(t, \zeta_l(t))(\tilde{a}(t, \zeta_l(t)) - a(t, \zeta_l(t)))\,\mathrm{d}t$$

$$(121) \qquad + \int_{\tilde{t}}^{\bar{t}} \hat{p}_1(t, \zeta_r(t))\Delta y(t, \zeta_r(t))(\tilde{a}(t, \zeta_r(t)) - a(t, \zeta_r(t)))\,\mathrm{d}t = O(\|\delta w\|_W^2)$$

In order to derive Y_l, we apply [23, Lemma 6.2.1] to IBVP on the truncated space-time cylinder $[\tilde{t}, \bar{t}] \times \Omega$ as in the proof of Lemma 11. We note that the term $O(\|\delta w\|_W^2)$ in (121) is uniform w.r.t. $\bar{\varepsilon}$.

Considering the first integral on the right hand side of (118), we can use (117) and Corollary 5 to show that

$$(122) \qquad \int_{\zeta_l(\tilde{t})}^{\zeta_r(\tilde{t})} \hat{p}_1(\tilde{t}, x) \Delta y(\tilde{t}, x) \, \mathrm{d}x = \int_{\zeta_l(\tilde{t})}^{\zeta_r(\tilde{t})} p_1(\tilde{t}, x) \Delta y(\tilde{t}, x) \, \mathrm{d}x + o(\|\delta w\|_W),$$

where $o(\|\delta w\|_W)$ is uniform w.r.t. $\bar{\varepsilon}$. We further obtain

$$\int_{\zeta_l(\tilde{t})}^{\zeta_r(\tilde{t})} p_1(\tilde{t}, x) \Delta y(\tilde{t}, x) \, \mathrm{d}x = \int_{\zeta_l(\tilde{t})}^{\xi_l(\tilde{t})} p_1(\tilde{t}, x) \Delta y(\tilde{t}, x) \, \mathrm{d}x$$

$$(123) \qquad + \int_{\xi_l(\tilde{t})}^{\xi_r(\tilde{t})} p_1(\tilde{t}, x) \Delta y(\tilde{t}, x) \, \mathrm{d}x + \int_{\xi_r(\tilde{t})}^{\zeta_r(\tilde{t})} p_1(\tilde{t}, x) \Delta y(\tilde{t}, x) \, \mathrm{d}x.$$

With the help of the local solutions $Y_{l/r}$ that we used to prove (121) and the boundedness of p_1, we can show the estimation

$$\left| \int_{\zeta_l(\tilde{t})}^{\xi_l(\tilde{t})} p_1(\tilde{t}, x) \Delta y(\tilde{t}, x) \, \mathrm{d}x \right| + \left| \int_{\xi_r(\tilde{t})}^{\zeta_r(\tilde{t})} p_1(\tilde{t}, x) \Delta y(\tilde{t}, x) \, \mathrm{d}x \right|$$

$$\le \left(|\xi_l(\tilde{t}) - \zeta_l(\tilde{t})| + |\xi_r(\tilde{t}) - \zeta_r(\tilde{t})| \right) \cdot O(\|\delta w|_W),$$

where the term $O(\|\delta w|_W)$ is uniform w.r.t. $\bar{\varepsilon}$. Using this result and (119), we can deduce that

$$(124) \qquad \left| \int_{\zeta_l(\tilde{t})}^{\xi_l(\tilde{t})} p_1(\tilde{t}, x) \Delta y(\tilde{t}, x) \, \mathrm{d}x \right| + \left| \int_{\xi_r(\tilde{t})}^{\zeta_l(\tilde{t})} p_1(\tilde{t}, x) \Delta y(\tilde{t}, x) \, \mathrm{d}x \right| \le \bar{\varepsilon} O(\|\delta w|_W).$$

Inserting (124) in (123), we obtain that (122) can be rewritten as

$$(125) \quad \int_{\zeta_l(\tilde{t})}^{\zeta_r(\tilde{t})} \hat{p}_1(\tilde{t}, x) \Delta y(\tilde{t}, x) \, \mathrm{d}x = \int_{\xi_l(\tilde{t})}^{\xi_r(\tilde{t})} p_1(\tilde{t}, x) \Delta y(\tilde{t}, x) \, \mathrm{d}x + \bar{\varepsilon} O(\|\delta w|_W) + o(\|\delta w\|_W).$$

Inserting (120), (121) and (125) in (97) yields

$$\frac{1}{[\bar{y}(\bar{t}, x_s(\bar{w}))]} \int_{x_s(\bar{w}) - \bar{\varepsilon}}^{x_s(\bar{w}) + \bar{\varepsilon}} \Delta y(\bar{t}, x) \, \mathrm{d}x = \int_{\xi_l(\tilde{t})}^{\xi_r(\tilde{t})} p_1(\tilde{t}, x) \Delta y(\tilde{t}, x) \, \mathrm{d}x$$

$$(126) \qquad + \int_{D_1} p_1(t, x) g_{u_1}(t, x, \bar{y}, \bar{u}_1) \delta u_1 \, \mathrm{d}x \, \mathrm{d}t + o(\|\delta w\|_W) + \bar{\varepsilon} O(\|\delta w\|_W).$$

We observe that due to [23, Lemma 6.3.5], p_1 coincides on D_1 with the adjoint state p defined in Definition 12 with enddata $p^{\bar{t}}(\cdot) = \mathbb{1}_{x_s(\bar{w})}(\cdot) \frac{1}{[\bar{y}(\bar{t}, x_s(\bar{w}))]}$. We will now have a closer look at the first term of the right hand side of (126). For all $\varepsilon \ge 0$ small enough it holds true that

$$\int_{\xi_l(\tilde{t})}^{\xi_r(\tilde{t})} p_1(\tilde{t}, x) \Delta y(\tilde{t}, x) \, \mathrm{d}x = \int_{\xi_l(\tilde{t})}^{f'(\bar{u}_m^{B,\mathfrak{a}}(\bar{t}_m^{\mathfrak{a}} - \varepsilon) \cdot (\tilde{t} - \bar{t}_m^{\mathfrak{a}} + \varepsilon)} p_1(\tilde{t}, x) \Delta y(\tilde{t}, x) \, \mathrm{d}x$$

$$(127) \qquad + \int_{f'(\bar{u}_m^{B,\mathfrak{a}}(\bar{t}_m^{\mathfrak{a}} - \varepsilon) \cdot (\tilde{t} - \bar{t}_m^{\mathfrak{a}} + \varepsilon)}^{\xi_r(\tilde{t})} p_1(\tilde{t}, x) \Delta y(\tilde{t}, x) \, \mathrm{d}x =: I_{1,\varepsilon} + I_{2,\varepsilon}.$$

In what follows, we choose ε as a function of δw satisfying

(128) $$\varepsilon(\delta w) \geq 2\|\delta w\|_W \quad \text{and} \quad \varepsilon(\delta w) \to 0 \ \text{if} \ \|\delta w\|_W \to 0.$$

Step 2: Considering $I_{1,\varepsilon}$, we note that since ξ_l ends in the inner of a rarefaction wave, for all $w \in B_\rho^W(\bar{w})$ with $\rho > 0$ small enough it holds true that

(129)
$$y(\tilde{t}, x, w) = \begin{cases} f'^{-1}\left(\frac{x}{\tilde{t}-t_m^{\mathfrak{a}}}\right) & \text{if } x \in]f'(u_{m+1}^{B,\mathfrak{a}}(t_m^{\mathfrak{a}})) \cdot (\tilde{t} - t_m^{\mathfrak{a}}), f'(u_m^{B,\mathfrak{a}}(t_m^{\mathfrak{a}})) \cdot (\tilde{t} - t_m^{\mathfrak{a}})[, \\ Y(\tilde{t}, x, w) & \text{if } x \in [f'(u_m^{B,\mathfrak{a}}(t_m^{\mathfrak{a}})) \cdot (\tilde{t} - t_m^{\mathfrak{a}}), f'(\bar{u}_m^{B,\mathfrak{a}}(\bar{t}_m^{\mathfrak{a}})) \cdot (\tilde{t} - \bar{t}_m^{\mathfrak{a}}) + \delta), \end{cases}$$

where $\delta > 0$ is a small constant and $w \in B_\rho^W(\bar{w}) \ni w \to Y(\tilde{t}, \cdot, w) \in C(I)$ is a smooth function obtained by [23, Lemma 6.2.7] with

$$I = (f'(\bar{u}_m^{B,\mathfrak{a}}(\bar{t}_m^{\mathfrak{a}})) \cdot (\tilde{t} - \bar{t}_m^{\mathfrak{a}}) - \delta, f'(\bar{u}_m^{B,\mathfrak{a}}(\bar{t}_m^{\mathfrak{a}})) \cdot (\tilde{t} - \bar{t}_m^{\mathfrak{a}}) + \delta).$$

We note that $y(\tilde{t}, x, w)$ is continuous in $x = f'(u_m^{B,\mathfrak{a}}(t_m^{\mathfrak{a}})) \cdot (\tilde{t} - t_m^{\mathfrak{a}})$ for all $w \in B_\rho^W(\bar{w})$.

Using a Taylor approximation of $f'(u_m^{B,\mathfrak{a}}(t_m^{\mathfrak{a}})) \cdot (\tilde{t} - t_m^{\mathfrak{a}})$ in $u_m^{B,\mathfrak{a}} = \bar{u}_m^{B,\mathfrak{a}}$ and $t_m^0 = \bar{t}_m^0$, we can deduce that there exists a constant $C > 0$ such that

$$\xi_l(\tilde{t}) < f'(\bar{u}_m^{B,\mathfrak{a}}(\bar{t}_m^{\mathfrak{a}})) \cdot (\tilde{t} - \bar{t}_m^{\mathfrak{a}}) - C\|\delta w\|_W \quad \text{and}$$

(130) $$[\xi_l(\tilde{t}), f'(\bar{u}_m^{B,\mathfrak{a}}(\bar{t}_m^{\mathfrak{a}})) \cdot (\tilde{t} - \bar{t}_m^{\mathfrak{a}}) - C\|\delta w\|_W] \subset [\xi_l(\tilde{t}), f'(u_m^{B,\mathfrak{a}}(t_m^{\mathfrak{a}})) \cdot (\tilde{t} - t_m^{\mathfrak{a}})]$$

hold for all $w \in B_\rho^W(\bar{w})$ with $\rho > 0$ small enough. Then (129) and (130) yield

(131)
$$\Delta y(\tilde{t}, x)\big|_{[\xi_l(\tilde{t}), f'(\bar{u}_m^{B,\mathfrak{a}}(\bar{t}_m^{\mathfrak{a}})) \cdot (\tilde{t} - \bar{t}_m^{\mathfrak{a}}) - C\|\delta w\|_W]} = \frac{x \cdot \delta t_m^{\mathfrak{a}}}{f''\big(f'^{-1}\big(\frac{x}{\tilde{t} - \bar{t}_m^{\mathfrak{a}}}\big)\big) \cdot (\tilde{t} - \bar{t}_m^{\mathfrak{a}})^2} + o(\|\delta w\|_W).$$

Using (131), we can rewrite the term $I_{1,\varepsilon}$ in (127) as follows:

$$\begin{aligned}
I_{1,\varepsilon} &= \int_{\xi_l(\tilde{t})}^{f'(\bar{u}_m^{B,\mathfrak{a}}(\bar{t}_m^{\mathfrak{a}})) \cdot (\tilde{t} - \bar{t}_m^{\mathfrak{a}}) - C\|\delta w\|_W} p_1(\tilde{t}, x) \frac{x \cdot \delta t_m^{\mathfrak{a}}}{f''\big(f'^{-1}\big(\frac{x}{\tilde{t} - \bar{t}_m^{\mathfrak{a}}}\big)\big) \cdot (\tilde{t} - \bar{t}_m^{\mathfrak{a}})^2} \, dx \\
&\quad + \int_{f'(\bar{u}_m^{B,\mathfrak{a}}(\bar{t}_m^{\mathfrak{a}})) \cdot (\tilde{t} - \bar{t}_m^{\mathfrak{a}}) - C\|\delta w\|_W}^{f'(\bar{u}_m^{B,\mathfrak{a}}(\bar{t}_m^{\mathfrak{a}} - \varepsilon)) \cdot (\tilde{t} - \bar{t}_m^{\mathfrak{a}} + \varepsilon)} p_1(\tilde{t}, x) \Delta y(\tilde{t}, x) \, dx + o(\|\delta w\|_W) \\
&= \int_{\xi_l(\tilde{t})}^{f'(\bar{u}_m^{B,\mathfrak{a}}(\bar{t}_m^{\mathfrak{a}})) \cdot (\tilde{t} - \bar{t}_m^{\mathfrak{a}})} p_1(\tilde{t}, x) \frac{x \cdot \delta t_m^{\mathfrak{a}}}{f''\big(f'^{-1}\big(\frac{x}{\tilde{t} - \bar{t}_m^{\mathfrak{a}}}\big)\big) \cdot (\tilde{t} - \bar{t}_m^{\mathfrak{a}})^2} \, dx \\
&\quad - \int_{f'(\bar{u}_m^{B,\mathfrak{a}}(\bar{t}_m^{\mathfrak{a}})) \cdot (\tilde{t} - \bar{t}_m^{\mathfrak{a}}) - C\|\delta w\|_W}^{f'(\bar{u}_m^{B,\mathfrak{a}}(\bar{t}_m^{\mathfrak{a}})) \cdot (\tilde{t} - \bar{t}_m^{\mathfrak{a}})} p_1(\tilde{t}, x) \frac{x \cdot \delta t_m^{\mathfrak{a}}}{f''\big(f'^{-1}\big(\frac{x}{\tilde{t} - \bar{t}_m^{\mathfrak{a}}}\big)\big) \cdot (\tilde{t} - \bar{t}_m^{\mathfrak{a}})^2} \, dx \\
&\quad + \int_{f'(\bar{u}_m^{B,\mathfrak{a}}(\bar{t}_m^{\mathfrak{a}})) \cdot (\tilde{t} - \bar{t}_m^{\mathfrak{a}}) - C\|\delta w\|_W}^{f'(\bar{u}_m^{B,\mathfrak{a}}(\bar{t}_m^{\mathfrak{a}} - \varepsilon)) \cdot (\tilde{t} - \bar{t}_m^{\mathfrak{a}} + \varepsilon)} p_1(\tilde{t}, x) \Delta y(\tilde{t}, x) \, dx + o(\|\delta w\|_W)
\end{aligned}$$

Due to the regularity of f and p_1, the second integral on the right hand side is equal to $o(\|\delta w\|_W)$. Using again the regularity of p_1, the choice of ε in (128) and the

representation of $y(\tilde{t}, x, w)$ in (129), which is in particular continuous, we obtain that the third integral is also equal to $o(\|\delta w\|_W)$. Hence, we can conclude

$$I_{1,\varepsilon} = \int_{\xi_l(\tilde{t})}^{f'(\bar{u}_m^{B,\mathfrak{a}}(\bar{t}_m^{\mathfrak{a}})) \cdot (\tilde{t} - \bar{t}_m^{\mathfrak{a}})} p_1(\tilde{t}, x) \frac{x \cdot \delta t_m^{\mathfrak{a}}}{f''(f'^{-1}(\frac{x}{\tilde{t} - \bar{t}_m^{\mathfrak{a}}}))(\tilde{t} - \bar{t}_m^{\mathfrak{a}})^2} \, dx + o(\|\delta w\|_W).$$

Since p_1 coincides with the adjoint p on D_1, we further obtain

$$I_{1,\varepsilon} = \int_{\xi_l(\tilde{t})}^{f'(\bar{u}_m^{B,\mathfrak{a}}(\bar{t}_m^{\mathfrak{a}})) \cdot (\tilde{t} - \bar{t}_m^{\mathfrak{a}})} p(\tilde{t}, x) \frac{x \cdot \delta t_m^{\mathfrak{a}}}{f''(f'^{-1}(\frac{x}{\tilde{t} - \bar{t}_m^{\mathfrak{a}}})) \cdot (\tilde{t} - \bar{t}_m^{\mathfrak{a}})^2} \, dx + o(\|\delta w\|_W)$$

$$= \int_{\xi_l(\tilde{t})/(\tilde{t} - \bar{t}_m^{\mathfrak{a}})}^{f'(\bar{u}_m^{B,\mathfrak{a}}(\bar{t}_m^{\mathfrak{a}}))} p(\tilde{t}, z(\tilde{t} - \bar{t}_m^{\mathfrak{a}})) \frac{z \cdot \delta t_m^{\mathfrak{a}}}{f''(f'^{-1}(z))} \, dz + o(\|\delta w\|_W)$$

$$(132) \qquad = \int_{f'(\bar{u}_{m+1}^{B,\mathfrak{a}}(\bar{t}_m^{\mathfrak{a}}))}^{f'(\bar{u}_m^{B,\mathfrak{a}}(\bar{t}_m^{\mathfrak{a}}))} \lim_{t \searrow \bar{t}_m^{\mathfrak{a}}} p(t, z(t - \bar{t}_m^{\mathfrak{a}})) \frac{z \cdot \delta t_m^{\mathfrak{a}}}{f''(f'^{-1}(z))} \, dz + o(\|\delta w\|_W).$$

The last equality holds due to the fact that for all $z \in [f'(\bar{u}_{m+1}^{B,\mathfrak{a}}(\bar{t}_m^{\mathfrak{a}})), f'(\bar{u}_m^{B,\mathfrak{a}}(\bar{t}_m^{\mathfrak{a}}))]$ the term $p(\cdot, z(\cdot - \bar{t}_m^{\mathfrak{a}}))$ is constant on $(\bar{t}_m^{\mathfrak{a}}, \tilde{t}]$ and is equal to zero for all

$$z \in [f'(\bar{u}_{m+1}^{B,\mathfrak{a}}(\bar{t}_m^{\mathfrak{a}})), \xi_l(\tilde{t})/(\tilde{t} - \bar{t}_m^{\mathfrak{a}})].$$

Step 3: In order to compute $I_{2,\varepsilon}$, we define suitable coefficients $\tilde{a}_{loc}, a_{loc}$ such that $\tilde{a}_{loc}|_{D_{2,\varepsilon}} \equiv \tilde{a}$ and $a_{loc}|_{D_{2,\varepsilon}} \equiv a$ hold for all $w \in B_\rho^W(\bar{w})$ with a sufficiently small $\rho > 0$ such that (100), (101) and (102) are satisfied. To this end, we choose some $\hat{t} \in (\bar{t}_{m-1}^{\mathfrak{a}}, \bar{t}_m^{\mathfrak{a}})$ and define $\tilde{a}_{loc}$ and a_{loc} on $[\hat{t}, \tilde{t}] \times \mathbb{R}$ by

$$\tilde{a}_{loc}(t, x) = M_{f'}, \text{ if } (t, x) \in [\hat{t}, \tilde{t}] \times [0, \xi_r(t)] \setminus D_{2, \frac{\varepsilon}{2}}, \qquad \tilde{a}_{loc}(t, x) = \tilde{a}(t, x), \text{ else}$$

$$a_{loc}(t, x) = M_{f'}, \text{ if } (t, x) \in [\hat{t}, \tilde{t}] \times [0, \xi_r(t)] \setminus D_2, \qquad a_{loc}(t, x) = a(t, x), \text{ else.}$$

Recalling the sets $D_{2,\varepsilon}$ and D_2 defined in (112) and (113), we choose the function $\varepsilon(\delta w)$ in (128) such that

$$(133) \qquad\qquad D_{2,\varepsilon} \subset D_{2, \frac{\varepsilon}{2}} \subset D_2 \quad \text{ for all } w \in B_\rho^W(\bar{w})$$

for sufficiently small $\rho > 0$.

Due to the construction of $\tilde{a}_{loc}$ and a_{loc} and the choice of ε in (128), one can show that (100), (101) and (102) hold on $[\hat{t}, \tilde{t}] \times \mathbb{R}$. Therefore,

$$(134) \qquad p_t + \tilde{a}_{loc} p_x = -\tilde{b}p \qquad \text{on } (\hat{t}, \tilde{t}) \times \mathbb{R}, \qquad p(\tilde{t}, \cdot) = p_1(\tilde{t}, \cdot) \qquad \text{on } \mathbb{R}$$

$$(135) \qquad \text{and } p_t + a_{loc} p_x = -bp \qquad \text{on } (\hat{t}, \tilde{t}) \times \mathbb{R}, \qquad p(\bar{t}, \cdot) = p_1(\tilde{t}, \cdot) \qquad \text{on } \mathbb{R}$$

admit reversible solutions $\hat{p}_2, p_2 \in C^{0,1}([\hat{t}, \tilde{t}] \times \mathbb{R})$, respectively, satisfying

$$(136) \qquad\qquad \hat{p}_2 \to p_2 \text{ in } C([\hat{t}, \tilde{t}] \times [-R, R]) \quad \text{if } \|\delta w\|_W \to 0$$

for all $R > 0$. We note that (133) yields that $\tilde{a}_{loc}(\cdot)$ coincides with $\tilde{a}(\cdot)$ on $D_{2,\varepsilon}$. Hence, if we consider (99) on $(\hat{t}, \tilde{t}) \times \mathbb{R}$, multiply it with $\hat{p}_2$ and apply integration by parts on

$D_{2,\varepsilon}$, then we obtain

$$I_{2,\varepsilon} = \int_{D_{2,\varepsilon}} \hat{p}_2(t,x)(g(t,x,\bar{y},u_1) - g(t,x,\bar{y},\bar{u}_1))\,\mathrm{d}x\,\mathrm{d}t + \int_0^{\xi_r(\hat{t})} \hat{p}_2(\hat{t},x)\Delta y(\hat{t},x)\,\mathrm{d}x$$

$$+ \int_{\bar{t}_m^{\mathfrak{a}}-\varepsilon}^{\tilde{t}} \hat{p}_2(t,\gamma(t,\varepsilon))\Delta y(t,\gamma(t,\varepsilon))(f'(\bar{u}_m^{B,\mathfrak{a}}(\bar{t}_m^{\mathfrak{a}} - \varepsilon)) - \tilde{a}(t,\gamma(t,\varepsilon)))\,\mathrm{d}t$$

$$- \int_{\hat{t}}^{\tilde{t}} \hat{p}_2(t,\xi_r(t))\Delta y(t,\xi_r(t))(f'(\bar{y}(t,\xi_r(t))) - \tilde{a}(t,\xi_r(t)))\,\mathrm{d}t$$

$$(137) \quad + \int_{\hat{t}}^{\bar{t}_m^{\mathfrak{a}}-\varepsilon} \hat{p}_2(t,0)\tilde{a}(t,0+)\Delta y(t,0+)\,\mathrm{d}t =: I_{20} + I_{21} + I_{22} + I_{23} + I_{24},$$

where $\gamma(t,\varepsilon) := f'(\bar{u}_m^{B,\mathfrak{a}}(\bar{t}_m^{\mathfrak{a}} - \varepsilon))(t - \bar{t}_m^{\mathfrak{a}} + \varepsilon)$.

Concerning (137), we note that $g(\cdot,y(w),u_1) = 0$ holds on $D_{2,0} \setminus D_{2,\varepsilon}$ for all $w \in B_\rho^W(\bar{w})$. From this result, the regularity of g and (136) we obtain

$$(138) \qquad I_{20} = (p_2(\cdot), g_{u_1}(\cdot,\bar{y},\bar{u}_1)\delta u_1)_{2,D_{2,0}} + o(\|\delta w\|_W).$$

Now, we show that

$$(139) \qquad \|I_{22}\| \leq C\|\delta w\|_W^2$$

and note that

$$(140) \qquad \|I_{23}\| \leq C\|\delta w\|_W^2$$

can be proved analogously.

To this end, we note that since the point $(\tilde{t}, f'(\bar{u}_m^{B,\mathfrak{a}}(\bar{t}_m^{\mathfrak{a}}))(\tilde{t} - \bar{t}_m^{\mathfrak{a}}))$ lies on the right boundary of the rarefaction wave emanating from $(\bar{t}_m^{\mathfrak{a}}, 0)$, [23, Lemma 6.2.7] implies that there exists a stripe S with

$$\left\{(t,x) \in [\bar{t}_m^{\mathfrak{a}},\tilde{t}] \times \Omega : x = f'(\bar{u}_m^{B,\mathfrak{a}}(\bar{t}_m^{\mathfrak{a}}))(t - \bar{t}_m^{\mathfrak{a}})\right\} \subset S$$

$$(141) \qquad \left\{(t,x) \in [\bar{t}_m^{\mathfrak{a}} - \delta,\tilde{t}] \times \Omega : x = M_{f'}(t - \bar{t}_m^{\mathfrak{a}} + \delta)\right\} \subset S,$$

and a continuously Fréchet-differentiable mapping $B_\rho^W(\bar{w}) \ni w \mapsto Y_m(\cdot;w) \in C(S)$ such that

$$(142) \qquad y(\cdot;w)|_{\hat{S}} = Y_m(\cdot;w) \text{ for all } w \in B_\rho^W(\bar{w})$$

holds true, where

$$(143) \qquad \hat{S} := \left\{(t,x) \in S : x > \max\left\{0, f'(u_m^{B,\mathfrak{a}}(\bar{t}_m^{\mathfrak{a}})) \cdot (t - \bar{t}_m^{\mathfrak{a}} + \frac{\varepsilon}{2})\right\}\right\}$$

provided that $\rho > 0$ and δ are small enough and $\tilde{t} > \bar{t}_m^{\mathfrak{a}}$ is sufficiently close to $\bar{t}_m^{\mathfrak{a}}$. Hereby, (141) implies that $\hat{S}$ is nonempty.

We observe that we can choose $\varepsilon > 0$ in (128) such that $(t,\gamma(t,\varepsilon)) \in \hat{S}$ holds true for all $t \in [\bar{t}_m^{\mathfrak{a}} - \varepsilon,\tilde{t}]$ and all $w \in B_\rho^W(\bar{w})$. Therefore, (142) yields for all $w \in B_\rho^W(\bar{w})$

$$\|\Delta y(t,\gamma(t,\varepsilon))\|_{C([\bar{t}_m^{\mathfrak{a}}-\varepsilon,\tilde{t}])} = \|Y_m(t,\gamma(t,\varepsilon);w) - Y_m(t,\gamma(t,\varepsilon);\bar{w})\|_{C([\bar{t}_m^{\mathfrak{a}}-\varepsilon,\tilde{t}])}$$

$$\leq \|Y_m(\cdot;w) - Y_m(\cdot;\bar{w})\|_{C(\hat{S})} = \|\frac{\partial}{\partial w}Y_m(\cdot;\bar{w})\|_{C(\hat{S})} \cdot \|\delta w\|_W + o(\|\delta w\|_W)$$

$$(144) \quad \leq C\|\delta w\|_W.$$

Next, we note that I_{22} in (137) can be estimated from above by

$$(145) \qquad |I_{22}| \le C \int_{\bar{t}_m^{\mathfrak{a}} - \varepsilon}^{\tilde{t}} |\Delta y(t, \gamma(t, \varepsilon))|^2 \, \mathrm{d}t,$$

where we have used the uniform boundedness of $\hat{p}_2$ and the regularity of f. From (144) and (145) we obtain that (139) holds true. Using similar arguments and (95), we obtain

$$(146) \qquad I_{24} = \big(p_2(\cdot, 0), f'(\bar{y}(\cdot, 0+))\delta u_m^{B,\mathfrak{a}}(\cdot)\big)_{2,[\hat{t},\bar{t}_m^{\mathfrak{a}}]} + o(\|\delta w\|_W).$$

Concerning I_{21}, from (136) and Corollary 5, we can deduce that

$$(147) \qquad I_{21} = \int_0^{\xi_r(\hat{t})} p_2(\hat{t}, x)\Delta y(\hat{t}, x) \, \mathrm{d}x + o(\|\delta w\|_W).$$

Due to [23, Lemma 6.3.5], p_2 and the adjoint state p coincide on $D_{2,0}$. Therefore, we can replace p_2 by p in the terms I_{20}, I_{21} and I_{24}.

Step 4: Next, we want to further simplify the term I_{21} in (147). To this end, we choose some $s \in (0, \varepsilon_g)$ and observe that since $\tilde{a}$ has no rarefaction centers on the domain $[s, \hat{t}] \times \mathbb{R}$, (100), (101) and (102) hold. Therefore,

$$(148) \qquad p_t + \tilde{a}p_x = -\tilde{b}p \qquad \text{on } (s, \hat{t}) \times \mathbb{R}, \qquad p(\hat{t}, \cdot) = p_2(\hat{t}, \cdot) \qquad \text{on } \mathbb{R}$$

$$(149) \qquad \text{and } p_t + ap_x = -bp \qquad \text{on } (s, \hat{t}) \times \mathbb{R}, \qquad p(\hat{t}, \cdot) = p_2(\hat{t}, \cdot) \qquad \text{on } \mathbb{R}$$

admit reversible solutions $\hat{p}_3, p_3 \in C^{0,1}([s, \hat{t}] \times \mathbb{R})$, respectively. Moreover,

$$(150) \qquad \hat{p}_3 \to p_3 \text{ in } C([s, \hat{t}] \times [-R, R]) \quad \text{if } \|\delta w\|_W \to 0$$

holds for all $R > 0$. Now, we consider (99) on $(s, \hat{t}) \times \Omega$, multiply it with $\hat{p}_3$ and apply integration by parts on D_3 defined in (114). This yields

$$\begin{aligned} I_{21} = & \int_{D_3} \hat{p}_3(t, x)(g(t, x, \bar{y}, u_1) - g(t, x, \bar{y}, \bar{u}_1)) \, \mathrm{d}x \, \mathrm{d}t \\ & + \int_s^{\hat{t}} \hat{p}_3(t, 0+)\tilde{a}(t, 0+)\Delta y(t, 0+)) \, \mathrm{d}t \\ & - \int_s^{\hat{t}} \hat{p}_3(t, \xi_r(t))\Delta y(t, \xi_r(t))(a(t, \xi_r(t)) - \tilde{a}(t, \xi_r(t))) \, \mathrm{d}t \\ & + \int_0^{\xi_r(s)} \hat{p}_3(s, x)\Delta y(s, x) \, \mathrm{d}x + o(\|\delta w\|_W) \\ (151) \qquad =: & \, I_{31} + I_{32} + I_{33} + I_{34} + o(\|\delta w\|_W). \end{aligned}$$

Analogously to the simplification of (137) in **Step 2**, one can also simplify the terms on the right hand side of (151). Using the regularity of g and (150), we obtain that

$$(152) \qquad I_{31} = \int_{D_3} p_3(t, x)g_{u_1}(t, x, \bar{y}, \bar{u}_1)\delta u_1 \, \mathrm{d}t \, \mathrm{d}x + o(\|\delta w\|_W).$$

Using that ξ_r ends in a point where the initial data is smooth and $\bar{w}$ satisfies (ND), we can use the same arguments as in the estimation of (139) and obtain

$$(153) \qquad I_{33} = O(\|\delta w\|_W^2).$$

Moreover, using (150) and Corollary 5, we obtain

$$
(154) \qquad I_{34} = \int_0^{\xi_r(s)} p_3(s,x)\Delta y(s,x)\,\mathrm{d}x + o(\|\delta w\|_W).
$$

Now we want to have a closer look at the term I_{32}. Observing that $\Delta y(\cdot, 0+) = \delta u_{B,\mathfrak{a}}$ holds on $[s,\hat{t}]$ due to (95), Lemma 4 yields that $\|\Delta y(\cdot, 0+)\|_{1,(s,\hat{t})} = O(\|\delta w\|_W)$. Using this result, (150) and the uniform boundedness of $\tilde{a}$, we can rewrite I_{32} by

$$
(155) \qquad I_{32} = \int_s^{\hat{t}} p_3(t,0)\tilde{a}(t,0+)\Delta y(t,0+)\,\mathrm{d}t + o(\|\delta w\|_W).
$$

Moreover, it holds true that

$$
\left\| \Delta y(\cdot, 0+) - \sum_{i=1}^{n_{t,\mathfrak{a}}+1} \delta u_i^{B,\mathfrak{a}}(\cdot)\mathbb{1}_{I_{B,\mathfrak{a}}^i(\bar{w})}(\cdot) - \sum_{i=1}^{n_{t,\mathfrak{a}}} \mathrm{sgn}(\delta t_i^{\mathfrak{a}})\mathbb{1}_{I(\bar{t}_i^{\mathfrak{a}},\bar{t}_i^{\mathfrak{a}}+\delta t_i^{\mathfrak{a}})}(\cdot)[\bar{u}_{B,\mathfrak{a}}(\bar{t}_i^{\mathfrak{a}})] \right\|_{1,(s,\hat{t})}
$$
$$
= o(\|\delta w\|_W).
$$

Using this result and the uniform boundedness of $\tilde{a}$ and p_3, we can rewrite the right hand side of (155) by

$$
\int_s^{\hat{t}} p_3(t,0+)\tilde{a}(t,0+)\Delta y(t,0+)\,\mathrm{d}t + o(\|\delta w\|_W)
$$
$$
= \sum_{i=1}^{n_{t,\mathfrak{a}}+1} \left(p_3(\cdot,0), \tilde{a}(t,0+)\delta u_i^{B,\mathfrak{a}}(\cdot) \right)_{2,I_{B,\mathfrak{a}}^i \cap [s,\hat{t}]}
$$
$$
+ \sum_{i \in I_{s,\mathfrak{a}},\, s < \bar{t}_i^{\mathfrak{a}} < \hat{t}} \left(p_3(\cdot,0), \tilde{a}(t,0+) \cdot \mathrm{sgn}(\delta t_i^{\mathfrak{a}})[\bar{u}_{B,\mathfrak{a}}(\bar{t}_i^{\mathfrak{a}})] \right)_{2,I(\bar{t}_i^{\mathfrak{a}},\bar{t}_i^{\mathfrak{a}}+\delta t_i^{\mathfrak{a}})} + o(\|\delta w\|_W)
$$
$$
(156) \quad =: I_{321} + I_{322} + o(\|\delta w\|_W).
$$

Since $\tilde{a}(\cdot, 0+) \to a(\cdot, 0+)$ in $L^1([s,\hat{t}])$ due to Corollary 5 and p_3 is bounded, we obtain that

$$
(157) \qquad I_{321} = \sum_{i=1}^{n_{t,\mathfrak{a}}+1} (p_3(\cdot,0), a(t,0+))\delta u_i^{B,\mathfrak{a}}(\cdot))_{2,I_{B,\mathfrak{a}}^i \cap [s,\hat{t}]} + o(\|\delta w\|_W).
$$

In order to simplify I_{322} we assume w.l.o.g. that $\delta t_i^{\mathfrak{a}} > 0$, where the case $\delta t_i^{\mathfrak{a}} < 0$ can be treated analogously and the case $\delta t_i^{\mathfrak{a}} = 0$ is trivial. We observe that for all $i \in I_{s,\mathfrak{a}}$ with $s < \bar{t}_i^{\mathfrak{a}} < \hat{t}$ it holds true that

$$
\|[\bar{u}_{B,\mathfrak{a}}(\bar{t}_i^{\mathfrak{a}})] - (u_i^{B,\mathfrak{a}}(\cdot) - \bar{u}_{i+1}^{B,\mathfrak{a}}(\cdot))\|_{L^1(\bar{t}_i^{\mathfrak{a}},\bar{t}_i^{\mathfrak{a}}+\delta t_i^{\mathfrak{a}})} = o(\|\delta w\|_W).
$$

This together with the uniform boundedness of $\tilde{a}$ and p_3 yields

$$
I_{322} = \sum_{i \in I_{s,\mathfrak{a}},\, s < \bar{t}_i^{\mathfrak{a}} < \hat{t}} \left(p_3(\cdot,0), \tilde{a}(\cdot,0+) \cdot (u_i^{B,\mathfrak{a}}(\cdot) - \bar{u}_{i+1}^{B,\mathfrak{a}}(\cdot)) \right)_{2,I(\bar{t}_i^{\mathfrak{a}},\bar{t}_i^{\mathfrak{a}}+\delta t_i^{\mathfrak{a}})} + o(\|\delta w\|_W)
$$
$$
(158) \quad = \sum_{i \in I_{s,\mathfrak{a}},\, s < \bar{t}_i^{\mathfrak{a}} < \hat{t}} \left(p_3(\cdot,0), f(u_i^{B,\mathfrak{a}}(\cdot)) - f(\bar{u}_{i+1}^{B,\mathfrak{a}}(\cdot)) \right)_{2,I(\bar{t}_i^{\mathfrak{a}},\bar{t}_i^{\mathfrak{a}}+\delta t_i^{\mathfrak{a}})} + o(\|\delta w\|_W),
$$

where the equality follows from the definition of $\tilde{a}$. We further observe that due to the regularity of p_3, f and $u_j^{B,\mathfrak{a}}$ for $j = 1, \ldots, n_{t,\mathfrak{a}}$ it holds true that

$$I_{322} = \sum_{i \in I_{s,\mathfrak{a}}, \, s < \bar{t}_i^{\mathfrak{a}} < \hat{t}} p_3(\bar{t}_i^{\mathfrak{a}}, 0)(f(\bar{u}_i^{B,\mathfrak{a}}(\bar{t}_i^{\mathfrak{a}})) - f(\bar{u}_{i+1}^{B,\mathfrak{a}}(\bar{t}_i^{\mathfrak{a}}))) \delta t_i^{\mathfrak{a}} + o(\|\delta w\|_W)$$

(159)
$$= \sum_{i \in I_{s,\mathfrak{a}}, \, s < \bar{t}_i^{\mathfrak{a}} < \hat{t}} p_3(\bar{t}_i^{\mathfrak{a}}, 0)[f(\bar{y}(\bar{t}_i^{\mathfrak{a}}, 0+))] \delta t_i^{\mathfrak{a}} + o(\|\delta w\|_W).$$

Inserting (157) and (159) in (156) and (156) in (155) finally yields

$$I_{32} = \int_s^{\hat{t}} p_3(t, 0+) a(t, 0+) \Delta y(t, 0+) \, \mathrm{d}t + o(\|\delta w\|_W)$$

$$= \sum_{i=1}^{n_{t,\mathfrak{a}}+1} (p_3(\cdot, 0), f'(\bar{u}_i^{B,\mathfrak{a}}(\cdot)) \delta u_i^{B,\mathfrak{a}}(\cdot))_{2, I_{B,\mathfrak{a}}^i \cap [s, \hat{t}]}$$

(160)
$$+ \sum_{i \in I_{s,\mathfrak{a}}, \, s < \bar{t}_i^{\mathfrak{a}} < \hat{t}} p_3(\bar{t}_i^{\mathfrak{a}}, 0)[f(\bar{y}(\bar{t}_i^{\mathfrak{a}}, 0+))] \delta t_i^{\mathfrak{a}} + o(\|\delta w\|_W).$$

Inserting (152), (153), (154) and (160) in (151) finally yields

$$I_{21} = (p_3(\cdot), g_{u_1}(\cdot, \bar{y}, \bar{u}_1) \delta u_1)_{2, D_3}$$

$$+ (p_3(s, \cdot), \Delta y(s, \cdot))_{2, (0, \xi_r(s))} + \sum_{i=1}^{n_{t,\mathfrak{a}}+1} (p_3(\cdot, 0), f'(\bar{u}_i^{B,\mathfrak{a}}) \delta u_i^{B,\mathfrak{a}})_{2, I_{B,\mathfrak{a}}^i \cap [s, \hat{t}]}$$

(161)
$$+ \sum_{i \in I_{s,\mathfrak{a}}, \, s \leq \bar{t}_i^{\mathfrak{a}} < \hat{t}} p_3(\bar{t}_i^{\mathfrak{a}}, 0)[f(\bar{y}(\bar{t}_i^{\mathfrak{a}}, 0+))] \delta t_i^{\mathfrak{a}} + o(\|\delta w\|_W).$$

We note that p_3 coincides with the adjoint state p on D_3 such that we obtain from (161)

$$I_{21} = (p(\cdot), g_{u_1}(\cdot, \bar{y}, \bar{u}_1) \delta u_1)_{2, D_3}$$

$$+ (p(s, \cdot), \Delta y(s, \cdot))_{2, (0, \xi_r(s))} + \sum_{i=1}^{n_{t,\mathfrak{a}}+1} (p(\cdot, 0), f'(\bar{u}_i^{B,\mathfrak{a}}) \delta u_i^{B,\mathfrak{a}})_{2, I_{B,\mathfrak{a}}^i \cap [s, \hat{t}]}$$

(162)
$$+ \sum_{i \in I_{s,\mathfrak{a}}, \, s \leq \bar{t}_i^{\mathfrak{a}} < \hat{t}} p(\bar{t}_i^{\mathfrak{a}}, 0)[f(\bar{y}(\bar{t}_i^{\mathfrak{a}}, 0+))] \delta t_i^{\mathfrak{a}} + o(\|\delta w\|_W).$$

The second term of the right side of (162) can be rewritten as follows:

$$(p(s, \cdot), \Delta y(s, \cdot))_{2, (0, \xi_r(s))}$$
$$= (p(s, \cdot), \Delta y(s, \cdot))_{2, (0, \bar{x}_l^0 - \varepsilon + f'(\bar{u}_l^0(\bar{x}_l^0 - \varepsilon)) \cdot s]}$$
$$+ (p(s, \cdot), \Delta y(s, \cdot))_{2, (\bar{x}_l^0 - \varepsilon + f'(\bar{u}_l^0(\bar{x}_l^0 - \varepsilon)) \cdot s, \bar{x}_l^0 + \varepsilon + f'(\bar{u}_{l+1}^0(\bar{x}_l^0 + \varepsilon)) \cdot s]}$$
$$+ (p(s, \cdot), \Delta y(s, \cdot))_{2, (\bar{x}_l^0 + \varepsilon + f'(\bar{u}_{l+1}^0(\bar{x}_l^0 + \varepsilon)) \cdot s, \xi_r(s))}$$

(163)
$$= I_{1,\varepsilon}^0 + I_{2,\varepsilon}^0 + I_{3,\varepsilon}^0.$$

Step 5: Using similar arguments as in the estimation of $I_{2,\varepsilon}$, one can firstly show that

$$I_{1,\varepsilon}^0 = (p(0, \cdot), \delta u_0(\cdot))_{2, (0, \bar{x}_l^0 - \varepsilon)} + o(\|\delta w\|_W),$$

where the term $o(\|\delta w\|_W)$ is uniform w.r.t. ε. Next, we observe that

$$
\left\| \delta u_0(\cdot) - \sum_{i=1}^{l} \delta u_i^0(\cdot) \mathbb{1}_{I_0^i(\bar{w})}(\cdot) - \sum_{i=1}^{l-1} \mathrm{sgn}(\delta x_i^0) \mathbb{1}_{I(\bar{x}_i^0, \bar{x}_i^0 + \delta x_i^0)}(\cdot) [\bar{u}_0(\bar{x}_i^0)] \delta x_i^0 \right\|_{1,(0,\bar{x}_l^0 - \varepsilon)}
$$
$$
= o(\|\delta w\|_W),
$$

where the term $o(\|\delta w\|_W)$ is uniform w.r.t. ε again. Hence, due to the regularity of p and the choice of ε in (128), it holds true that

$$
I_{1,\varepsilon}^0 = \sum_{i=1}^{l} (p(0,\cdot), \delta u_i^0(\cdot))_{2, I_0^i(\bar{w}) \cap [0,\xi_r(0)]} + \sum_{i=1}^{l-1} \mathbb{1}_{[0,\xi_r(0)]}(\bar{x}_i^0) p(0, \bar{x}_i^0)[\bar{u}_0(\bar{x}_i^0)] \delta x_i^0
$$
$$
+ o(\|\delta w\|_W)
$$
$$
(164) \qquad = \sum_{i=1}^{l} (p(0,\cdot), \delta u_i^0(\cdot))_{2, I_0^i(\bar{w})} + \sum_{i=1}^{l-1} p(0, \bar{x}_i^0)[\bar{u}_0(\bar{x}_i^0)] \delta x_i^0 + o(\|\delta w\|_W),
$$

where the second equality holds due to the fact that the adjoint state p is by definition equal to zero on D_- and on $\{(t,x) \in [0,\bar{t}] \times [0,\mathfrak{b}] : x > \xi_r(t)\}$. Analogously, one can show that

$$
(165) \qquad I_{3,\varepsilon}^0 = \sum_{i=l+1}^{n_x+1} (p(0,\cdot), \delta u_i^0(\cdot))_{2, I_0^i(\bar{w})} + \sum_{i=l+1}^{n_x} p(0, \bar{x}_i^0)[\bar{u}_0(\bar{x}_i^0)] \delta x_i^0 + o(\|\delta w\|_W).
$$

Similar to the estimation of $I_{1,\varepsilon}$, using that

$$
y(s,x,w)\big|_{]f'(u_l^0(x_l^0)) \cdot s + x_l^0, f'(u_{l+1}^0(x_l^0)) \cdot s + x_l^0[} = f'^{-1}\left(\frac{x - x_l^0}{s} \right) \qquad \forall w \in B_\rho^W(\bar{w})
$$

for sufficiently small $\rho > 0$, we can further show that

$$
(166) \qquad I_{2,\varepsilon}^0 = -\int_{f'(\bar{u}_l^0(\bar{x}_l^0))}^{f'(\bar{u}_{l+1}^0(\bar{x}_l^0))} \lim_{t \searrow 0} p(t, zt + \bar{x}_l^0) \frac{\delta x_l^0}{f''(f'^{-1}(z))} \, \mathrm{d}z + o(\|\delta w\|_W),
$$

where the term $o(\|\delta w\|_W)$ is uniform w.r.t. ε. Inserting (164), (165) and (166) in (163), we obtain

$$
(p(s,\cdot), \Delta y(s,\cdot))_{2,(0,\xi_r(s)]} = \sum_{i=1}^{n_x+1} (p(0,\cdot), \delta u_i^0(\cdot))_{2, I_0^i(\bar{w})}
$$
$$
- \int_{f'(\bar{u}_l^0(\bar{x}_l^0))}^{f'(\bar{u}_{l+1}^0(\bar{x}_l^0))} \lim_{t \searrow 0} p(t, zt + \bar{x}_l^0) \frac{\delta x_l^0}{f''(f'^{-1}(z))} \, \mathrm{d}z
$$
$$
(167) \qquad + \sum_{i \in I_{s,0}(\bar{w})} p(0, \bar{x}_i^0)[\bar{u}_0(\bar{x}_i^0)] \delta x_i^0 + o(\|\delta w\|_W).
$$

Inserting (167) in (162), we can deduce

$$
\begin{aligned}
I_{21} = {}& (p(\cdot), g_{u_1}(\cdot, y, u_1)\delta u_1)_{2,D_3} + \sum_{i=1}^{n_x+1} (p(0,\cdot), \delta u_i^0(\cdot))_{2,I_0^i(\bar{w})} \\
& - \int_{f'(\bar{u}_l^0(\bar{x}_l^0))}^{f'(\bar{u}_{l+1}^0(\bar{x}_l^0))} \lim_{t\searrow 0} p(t, zt + \bar{x}_l^0) \frac{\delta x_l^0}{f''(f'^{-1}(z))} \, dz \\
& + \sum_{i\in I_{s,0}(\bar{w})} p(0,\bar{x}_i^0)[\bar{u}_0(\bar{x}_i^0)]\delta x_i^0 + \sum_{i=1}^{n_{t,a}+1} (p(\cdot,0), f'(\bar{u}_i^{B,\mathfrak{a}}(\cdot))\delta u_i^{B,\mathfrak{a}}(\cdot))_{2,I_{B,\mathfrak{a}}^i(\bar{w})\cap[0,\hat{t}]}
\end{aligned}
$$

$$
(168) \quad + \sum_{i\in I_{s,\mathfrak{a}}(\bar{w}),\, s\le \bar{t}_i^{\mathfrak{a}}<\hat{t}} p(\bar{t}_i^{\mathfrak{a}},0)[f(\bar{y}(\bar{t}_i^{\mathfrak{a}},0+))]\delta t_i^{\mathfrak{a}} + o(\|\delta w\|_W).
$$

As already mentionned at the end of step 2, p_2 and the adjoint state p coincide on $D_{2,0}$ such that we can replace p_2 by p in (138) and (146) yielding

$$
(169) \qquad\qquad I_{20} = (p(\cdot), g_{u_1}(\cdot, \bar{y}, \bar{u}_1)\delta u_1)_{2,D_{2,0}} + o(\|\delta w\|_W)
$$

$$
(170) \qquad \text{and} \;\; I_{24} = \big(p(\cdot,0), f'(\bar{y}(\cdot,0+))\delta u_m^{B,\mathfrak{a}}(\cdot)\big)_{2,[\hat{t},\bar{t}_m^{\mathfrak{a}}]} + o(\|\delta w\|_W).
$$

Inserting (139), (140), (168), (169) and (170) in (137), we obtain

$$
\begin{aligned}
I_{2,\varepsilon} = {}& (p(\cdot), g_{u_1}(\cdot, \bar{y}, \bar{u}_1)\delta u_1)_{2,D_{2,0}\cup D_3} \\
& + \sum_{i=1}^{n_{t,a}+1} (p(\cdot,0), f'(\bar{u}_i^{B,\mathfrak{a}}(\cdot))\delta u_i^{B,\mathfrak{a}}(\cdot))_{2,I_{B,\mathfrak{a}}^i(\bar{w})\cap[0,\bar{t}_m^{\mathfrak{a}}]} \\
& + \sum_{i\in I_{s,\mathfrak{a}}(\bar{w}),\, s\le \bar{t}_i^{\mathfrak{a}}<\hat{t}} p(\bar{t}_i^{\mathfrak{a}},0)[f(\bar{y}(\bar{t}_i^{\mathfrak{a}},0+))]\delta t_i^{\mathfrak{a}} + \sum_{i=0}^{n_x+1} (p(0,\cdot), \delta u_i^0(\cdot))_{2,I_0^i(\bar{w})}. \\
& - \int_{f'(\bar{u}_l^0(\bar{x}_l^0))}^{f'(\bar{u}_{l+1}^0(\bar{x}_l^0))} \lim_{t\searrow 0} p(t, zt + \bar{x}_l^0) \frac{\delta x_l^0}{f''(f'^{-1}(z))} \, dz
\end{aligned}
$$

$$
(171) \qquad + \sum_{i\in I_{s,0}(\bar{w})} p(0,\bar{x}_i^0)[\bar{u}_0(\bar{x}_i^0)]\delta x_i^0 + o(\|\delta w\|_W).
$$

Since $p(\cdot,0)|_{(\bar{t}_m^{\mathfrak{a}},\bar{t}]} \equiv 0$, choosing $s>0$ sufficiently small, we obtain that (171) can be simplified to

$$
\begin{aligned}
I_{2,\varepsilon} = {}& (p(\cdot), g_{u_1}(\cdot, \bar{y}, \bar{u}_1)\delta u_1)_{2,D_{2,0}\cup D_3} \\
& + \sum_{i=1}^{n_{t,a}+1} (p(\cdot,0), f'(\bar{u}_i^{B,\mathfrak{a}}(\cdot))\delta u_i^{B,\mathfrak{a}}(\cdot))_{2,I_{B,\mathfrak{a}}^i(\bar{w})\cap[0,\bar{t}]} \\
& + \sum_{i\in I_{s,\mathfrak{a}}(\bar{w}),\, \bar{t}_i^{\mathfrak{a}}\le\bar{t}} p(\bar{t}_i^{\mathfrak{a}},0)[f(\bar{y}(\bar{t}_i^{\mathfrak{a}},0+))]\delta t_i^{\mathfrak{a}} + \sum_{i=0}^{n_x+1} (p(0,\cdot), \delta u_i^0(\cdot))_{2,I_0^i(\bar{w})}. \\
& - \int_{f'(\bar{u}_l^0(\bar{x}_l^0))}^{f'(\bar{u}_{l+1}^0(\bar{x}_l^0))} \lim_{t\searrow 0} p(t, zt + \bar{x}_l^0) \frac{\delta x_l^0}{f''(f'^{-1}(z))} \, dz
\end{aligned}
$$

$$
(172) \qquad + \sum_{i\in I_{s,0}(\bar{w})} p(0,\bar{x}_i^0)[\bar{u}_0(\bar{x}_i^0)]\delta x_i^0 + o(\|\delta w\|_W).
$$

Inserting (172) and (132) in (127) and further (127) in (126) yields

$$\frac{1}{[\bar{y}(\bar{t}, x_s(\bar{w}))]} \int_{x_s(\bar{w})-\bar{\varepsilon}}^{x_s(\bar{w})+\bar{\varepsilon}} \Delta y(\bar{t}, x)\, \mathrm{d}x = \int_{D_{2,0}\cup D_3\cup D_1} p(t,x) g_{u_1}(t, x, \bar{y}, \bar{u}_1)\delta u_1\, \mathrm{d}x\, \mathrm{d}t$$

$$+ \int_{f'(\bar{u}_{m+1}^{B,\mathfrak{a}}(\bar{t}_m^{\mathfrak{a}}))}^{f'(\bar{u}_m^{B,\mathfrak{a}}(\bar{t}_m^{\mathfrak{a}}))} \lim_{t\searrow \bar{t}_m^{\mathfrak{a}}} p(t, z(t-\bar{t}_m^{\mathfrak{a}}))\frac{\delta t_m^{\mathfrak{a}}\cdot z}{f''(f'^{-1}(z))}\, \mathrm{d}z$$

$$+ \sum_{i=1}^{n_{t,\mathfrak{a}}+1} (p(\cdot, 0), f'(u_i^{B,\mathfrak{a}}(\cdot))\delta u_i^{B,\mathfrak{a}}(\cdot))_{2, I_{B,\mathfrak{a}}^i(\bar{w})\cap[0,\bar{t}]} + \sum_{i\in I_{s,\mathfrak{a}},\, \bar{t}_i^{\mathfrak{a}}\leq\bar{t}} p(\bar{t}_i^{\mathfrak{a}}, 0)[f(\bar{y}(t_i, 0+))]\delta t_i^{\mathfrak{a}}$$

$$+ \sum_{i=0}^{n_x+1} (p(0,\cdot), \delta u_i^0(\cdot))_{2, I_0^i} + \sum_{i\in I_{s,0}} p(0, \bar{x}_i^0)[\bar{u}_0(\bar{x}_i^0)]\delta x_i^0$$

$$- \int_{f'(\bar{u}_l^0(\bar{x}_l^0))}^{f'(\bar{u}_{l+1}^0(\bar{x}_l^0))} \lim_{t\searrow 0} p(t, zt+\bar{x}_l^0)\frac{\delta x_l^0}{f''(f'^{-1}(z))}\, \mathrm{d}z + o(\|\delta w\|_W) + \bar{\varepsilon} O(\|\delta w\|_W).$$

Here, we have used that p_1 coincides with the adjoint state p on D_1. Setting $\bar{\varepsilon} := \bar{\varepsilon}(\delta w)$ such that $\bar{\varepsilon}(\delta w) \to 0$ if $\|\delta w\|_W \to 0$ and using (96), we can finally deduce Fréchet-differentiability of the mapping (11) with derivative

$$\frac{\partial}{\partial w} x_s(w)\cdot\delta w = \int_{D_{2,0}\cup D_3\cup D_1} p(t,x) g_{u_1}(t, x, \bar{y}, \bar{u}_1)\delta u_1\, \mathrm{d}x\, \mathrm{d}t$$

$$+ \int_{f'(\bar{u}_{m+1}^{B,\mathfrak{a}}(\bar{t}_m^{\mathfrak{a}}))}^{f'(\bar{u}_m^{B,\mathfrak{a}}(\bar{t}_m^{\mathfrak{a}}))} \lim_{t\searrow \bar{t}_m^{\mathfrak{a}}} p(t, z(t-\bar{t}_m^{\mathfrak{a}}))\frac{\delta t_m^{\mathfrak{a}}\cdot z}{f''(f'^{-1}(z))}\, \mathrm{d}z$$

$$+ \sum_{i=1}^{n_{t,\mathfrak{a}}+1} (p(\cdot, 0), f'(u_i^{B,\mathfrak{a}}(\cdot))\delta u_i^{B,\mathfrak{a}}(\cdot))_{2, I_{B,\mathfrak{a}}^i(\bar{w})\cap[0,\bar{t}]}$$

$$+ \sum_{i\in I_{s,\mathfrak{a}},\, \bar{t}_i^{\mathfrak{a}}\leq\bar{t}} p(\bar{t}_i^{\mathfrak{a}}, 0)[f(\bar{y}(t_i, 0+))]\delta t_i^{\mathfrak{a}}$$

$$+ \sum_{i=0}^{n_x+1} (p(0,\cdot), \delta u_i^0(\cdot))_{2, I_0^i} + \sum_{i\in I_{s,0}} p(0, \bar{x}_i^0)[\bar{u}_0(\bar{x}_i^0)]\delta x_i^0$$

(173)
$$- \int_{f'(\bar{u}_l^0(\bar{x}_l^0))}^{f'(\bar{u}_{l+1}^0(\bar{x}_l^0))} \lim_{t\searrow 0} p(t, zt+\bar{x}_l^0)\frac{\delta x_l^0}{f''(f'^{-1}(z))}\, \mathrm{d}z.$$

Since it holds $\mathrm{supp}_x(p) \subset D_{2,0}\cup D_3\cup D_1$, we can finally conclude that the representation in (173) coincides with the representation in Theorem 13. In the case of $\tilde{n}$ rarefaction center $\{i_1, \ldots, i_{\tilde{n}}\}$, the term $(p(s,\cdot), \Delta y(s,\cdot))_{2,\Omega}$ in (163) is splitted as follows:

$$(p(s,\cdot), \Delta y(s,\cdot))_{2,\Omega} = (p(s,\cdot), \Delta y(s,\cdot))_{2, (-\infty, \bar{x}_{i_1}^0-\varepsilon+f'(\bar{u}_{i_1}^0(\bar{x}_{i_1}^0-\varepsilon))\cdot s]}$$

$$+ \sum_{j=2}^{\tilde{n}-1} (p(s,\cdot), \Delta y(s,\cdot))_{2, (\bar{x}_{i_j}^0-\varepsilon+f'(\bar{u}_{i_j}^0(\bar{x}_{i_j}^0-\varepsilon))\cdot s, \bar{x}_{i_j}^0+\varepsilon+f'(\bar{u}_{i_j+1}^0(\bar{x}_{i_j}^0+\varepsilon))\cdot s]}$$

$$+ (p(s,\cdot), \Delta y(s,\cdot))_{2, (\bar{x}_{i_{\tilde{n}}}^0+\varepsilon+f'(\bar{u}_{i_{\tilde{n}}+1}^0(\bar{x}_{i_{\tilde{n}}}^0+\varepsilon))\cdot s, \infty)} = \sum_{k=1}^{\tilde{n}+1} I_{k,\varepsilon}^0.$$

Then we can treat all terms $I_{k,\varepsilon}^0$ analogously to $I_{1,\varepsilon}^0$, $I_{2,\varepsilon}^0$ and $I_{3,\varepsilon}^0$, respectively and obtain the desired result. In the case that also the right boundary is within in shock

funnel and there are several rarefaction centers, we introduce time points $\tilde{t}_i > \bar{t}_i^{\mathfrak{a}}$ for $i \in I_{r,\mathfrak{a}}(\bar{w})$ and $\tilde{t}_i > \bar{t}_i^{\mathfrak{b}}$ for $i \in I_{r,\mathfrak{b}}(\bar{w})$ satisfying (110), respectively. The treatment of the case that $|\mathbb{T}^{\mathfrak{a}}|, |\mathbb{T}^{\mathfrak{b}}| > 1$ is basically the same as it was shown in Step 3.   □

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