The isothermal Euler equations for ideal gas with source term: Product solutions, flow reversal and no blow up

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Abstract

The one–dimensional isothermal Euler equations are a well-known model for the flow of gas through a pipe. An essential part of the model is the source term that models the influence of gravity and friction on the flow. In general the solutions of hyperbolic balance laws can blow-up in finite time. We show the existence of initial data with arbitrarily large C^1–norm of the logarithmic derivative where no blow up in finite time occurs. The proof is based upon the explicit construction of product solutions. Often it is desirable to have such analytical solutions for a system described by partial differential equations, for example to validate numerical algorithms, to improve the understanding of the system and to study the effect of simplifications of the model. We present solutions of different types: In the first type of solutions, both the flow rate and the density are increasing functions of time. We also present a second type of solutions where on a certain time interval, both the flow rate and the pressure decrease.

In pipeline networks, the bi-directional use of the pipelines is sometimes desirable. In this paper we present a classical solution of the isothermal Euler equations where the direction of the gas flow changes. In the solution, at the time before the direction of the flow is reversed, the gas flow rate is zero everywhere in the pipe.

Keywords: Isothermal Euler equations, global classical solutions, no blow up, ideal gas, bi-directional flow, product solutions, flow reversal, transient flow, classical solutions, transsonic flow

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1. Introduction

Pipeline networks for gas transportation are an important part of our infrastructure. The one–dimensional isothermal Euler equation are a well–established model for the flow of gas through pipes, see [1], [17]. We consider the Euler equations for ideal gas with a non-vanishing source term that models the influence of the friction and the pipeline slope.

The corresponding stationary states have been studied in [7]. In particular, it has been shown that due to the friction term for nonzero flow rates the stationary states exist as classical solution only on a finite space interval until they break down at a certain critical length in the direction of the flow. In this paper we construct explicit solutions that illustrate that for certain instationary initial states no such blow-up occurs. The solutions exist as a transient classical solutions for all times $t > 0$ on the whole real line. These solutions are product solutions of the partial differential equations for ideal gas, where one factor in the solution is a function of time and the other factor depends on the space variable only. The presented solutions are useful reference solutions that help to improve the understanding of the system and to test numerical algorithms for the solution of the partial differential equations. In [15], the existence of global smooth solutions for a model equation for fluid flow in a pipe is shown under smallness assumptions for the initial state. In [13], [14] special explicit solutions are constructed to study the blow up behavior of the multi-dimensional Euler equations for compressible fluid flow without source term. Spherically symmetric solutions of the compressible Euler equations are studied in [4]. A class of analytical solutions with shocks to the Euler equations with source terms has also been presented in [5], [6]. Traveling waves solutions and self-similar solutions for the one-dimensional compressible Euler equations with heat conduction and without source term have been studied in [2]. In this paper, we consider classical solutions for the one-dimensional isothermal Euler equations with a source term. In our solutions, several types of monotonicity behavior appear:

1. The first type of behavior is a solution where both the absolute value of the flow rate and the density are strictly increasing functions of time at each point in space (see Lemma 1).
2. Also the following monotonicity behavior occurs: On a certain bounded time interval, both the absolute value of the flow rate and the pressure decrease as a function of time everywhere in space. At the end of this time interval, the flow rate is zero. Then the direction of the flow is reversed and the monotonicity type of the solution changes again to the type described in the previous point

The second type of solutions is particularly interesting since in pipeline networks sometimes the direction of the gas flow is not obvious. An example of such a network is presented in [7]. The bi-directional use of natural gas in certain pipelines occurs as supply and demand change with time and cause reversals in gas flow. We present a global classical solution of the one–dimensional isothermal Euler equation where the direction of the gas flow is reversed.

This paper has the following structure. In Section 2, the system of balance laws that we consider is stated. In Section 3, a no blow up result and a result on the change of the direction of the flow are stated as theorems and proved.

Product solutions where the factor that depends on the space variable only is given by an exponential function are discussed.

At the end of the paper in Section 4 we also study the effect of model simplifications: If the quasilinear equations are simplified to semilinear partial differential equations by omitting certain nonlinear space derivative terms, the parameters in the product solution change but it still keeps the same form. However, if the model is further simplified by omitting also a time-derivative, the nature of the solution changes completely, since the new model only allows for product solutions given by exponentials that are also travelling waves.

We want to emphasize that we construct solutions for the *one–dimensional* isothermal compressible Euler equations. This model is important as a model for the gas transport in pipelines. However, it is not clear how the constructions can be generalized to the case of multi–dimensional spaces.

2. The model: A system of balance laws

Let $D_{pipe} > 0$ denote the diameter of the pipe, $\lambda_{fric} > 0$ the friction coefficient and $\alpha \in (-\infty, \infty)$ the slope of the pipe. Define $z^e = \sin(\alpha)$ and $\theta = \frac{\lambda_{fric}}{D_{pipe}}$. Let g denote the gravitational constant and let $a > 0$ denote the sound speed in the gas. We assume that a is constant, that is we consider ideal gas. We study the isothermal Euler equations (see [1], [8])

$$\begin{cases} \rho_t + q_x = 0, \\ q_t + \left(\frac{q^2}{\rho} + a^2 \rho\right)_x = -\frac{1}{2}\theta \frac{q\,|q|}{\rho} - \rho\, g\, z^e \end{cases} \tag{1}$$

for $t \geq t_0$ and $x \in (-\infty, \infty)$. At the time $t = t_0$, an initial state is prescribed. In the model, ρ denotes the gas density and q denotes the flow rate. Let us recall that the Mach number M is defined as

$$M = \frac{q}{a\,\rho}.$$

A state is called subsonic if $|M| < 1$.

3. A no blow up result and change of the direction of the flow

In the theory of semi-global solutions for quasilinear hyperbolic systems (see for example [12]), the existence of classical solutions on a given finite time interval is shown for initial data with sufficiently small C^1-norm. This type of results is the foundation for results of global exact controllability with classical solutions, see for example [9]. The global existence of classical solutions for sufficiently small smooth initial data is shown in [19] for a friction term that is simpler than in (1). If the C^1-norm of the initial data is too large, in general shocks can develop after finite time such that the classical solution breaks down after finite time. However, for some initial data the solution exists as a classical solution for all $t \geq 0$.

Example 1. *Let a real number* $\rho_0 > 0$ *be given. For* $t \geq 0$ *and* $x \in (-\infty, \infty)$ *define* $\rho(t, x) = \rho_0$ *and for a real parameter* $P_0 > 0$ *define*

$$q(t, x) = \frac{1}{P_0 + \frac{1}{2}\frac{\theta}{\rho_0}t}. \tag{2}$$

Then for $z^e = 0$*, that is for a horizontal pipe,* (ρ, q) *solves (1). The solution exists for all* $t \geq 0$ *and for all* $x \in (-\infty, \infty)$ *we have* $\lim_{t\to\infty} q(t, x) = 0$ *and* $\lim_{t\to\infty} q_t(t, x) = 0$*. In particular, the solution is bounded.*

In Example 1, the spatial derivative of the state is zero.

Example 2. *System (1) has also solutions of travelling wave type. For a real number* λ*, let*

$$(q(t,x), \rho(t,x)) = (\lambda\, \alpha(\lambda\, t - x),\ \alpha(\lambda\, t - x)) \tag{3}$$

where the function α *is given by*

$$\alpha(z) = C\, \exp\left(\frac{\lambda\, |\lambda|\, \theta + 2\, g\, z^e}{2\, a^2} z\right) \tag{4}$$

and $C > 0$ *is a positive constant. Note that here the quotient* $\frac{\rho_x}{\rho}$ *is constant.*

In Theorem 1 we show that for initial data with arbitrarily large C^1-norm and arbitrarily large logarithmic derivative the solution does not necessarily break down after finite time and a global classical solution can exist. We prove the following no blow-up result:

Theorem 1. *For all* $K \in \{1, 2, 3, ...\}$ *there exists a continuously differentiable initial state* $(\rho_0(x), q_0(x))$ *(for* $x \in (-\infty, \infty)$*) with* $\rho_0(x) > 0$*,* $\rho_0(0) = 1$ *and*

$$|\rho_0'(0)| \geq K \tag{5}$$

that generates a global classical solution of (1) for all $t > 0$*,* $x \in (-\infty, \infty)$ *such that the* C^1*-norm of the solution does not blow up in finite time and the* C^1*-norm of the corresponding Mach number remains uniformly bounded on the whole time-space domain* $[0, \infty) \times (-\infty, \infty)$*.*

Remark 1. *Instead of the normalization* $\rho_0(0) = 1$ *we can also assume that the logarithmic derivative satisfies*

$$\frac{|\rho_0'(0)|}{|\rho_0(0)|} \geq K \tag{6}$$

A condition of this type is included in Theorem 1 on account of the following property of the Euler equations: If (ρ, q) *solves (1), then for each real number* $\lambda > 0$ *also the function*

$$(\lambda\, \rho,\ \lambda q)$$

solves (1). Condition (6) is invariant to such a rescaling.

Remark 2. *In order to obtain classical solutions of initial boundary value problems for a finite space interval* $[a, b]$ *for example corresponding to a finite pipe of length* $L = b - a$ *the boundary traces of the state at* $x = a$ *and* $x = b$ *can be used to obtain the necessary boundary values in the boundary conditions. For example, at* $x = a$ *the value of* $\rho(t, a)$ *and at* $x = b$ *the value of* $q(t, b)$ *can be prescribed. In this way it can be shown that also for initial and boundary data with arbitrarily large derivatives global classical solutions can exist.*

Remark 3. *The proof of Theorem 1 is based upon an explicit construction of a product solution that is given in Lemma 1. A certain drawback of this solution is that while the Mach number remains uniformly bounded with respect to time, both* ρ_0 *and* q_0 *are strictly increasing unbounded functions of* x *and* t*. Note however, that the solution is smooth.*

The following result states that for certain initial states of the system there exists a finite time t_1 such that after the time t_1 the direction of the flow changes.

Theorem 2. *There exists an initial state* $(q(t_0, x), \rho(t_0, x))$ *with* $q(t_0, x) > 0$ *for all* $x \in (-\infty, \infty)$ *and a time* $t_1 > t_0$ *such that (1) has a global classical solution where after the finite time* t_1*, the flow changes its direction, that is* $q(t, x) < 0$ *for all* $t > t_1$*,* $x \in (-\infty, \infty)$*.*

More precisely, assume that $z^e < 0$*. For* $\beta = -\frac{g\, z^e}{a^2} > 0$ *the state* $\bar{q}(t, x) = 0$*,* $\bar{\rho}(t, x) = \exp(\beta\, x)$ *is a stationary solution of (1) that is defined for all* $x \in (-\infty, \infty)$*,* $t \geq 0$*.*

For $\beta > -\frac{g\, z^e}{a^2} > 0$*, define* $\omega = \frac{\sqrt{\theta}}{\sqrt{2}} \sqrt{|a^2\, \beta + g\, z^e|} \neq 0$ *and* $t_1 = \frac{\pi}{2} \frac{1}{\omega}$*. For* $t \in (0, t_1]$ *and* $x \in (-\infty, \infty)$ *let*

$$q(t, x) = \frac{2\, \omega}{\theta} \cot(\omega\, t) \sin(\omega\, t)^{-\frac{2\,|\beta|}{\theta}} \exp(\beta\, x), \tag{7}$$

$$\rho(t, x) = \sin(\omega\, t)^{-\frac{2\,|\beta|}{\theta}} \exp(\beta\, x). \tag{8}$$

Then (ρ, q) *is a classical solution of (1) on the time-interval* $(0, t_1]$ *for* $x \in (-\infty, \infty)$*. It is the unique classical solution of (1) that satisfies the conditions*

$$q(t_1, x) = 0, \quad \rho(t_1, x) = \exp(\beta\, x). \tag{9}$$

Moreover, for $t \geq t_1$*, the unique classical solution of (1) that satisfies the initial conditions (9) is given by the equations*

$$q(t, x) = -\frac{2\, \omega}{\theta} \tanh(\omega\, (t - t_1)) \cosh(\omega\, (t - t_1))^{\frac{2\,|\beta|}{\theta}} \exp(\beta\, x), \tag{10}$$

$$\rho(t, x) = \cosh(\omega\, (t - t_1))^{\frac{2\,|\beta|}{\theta}} \exp(\beta\, x). \tag{11}$$

Hence the direction of the flow changes.

For $t \in (0, t_1]$*, the Mach number is given by* $M(t) = \frac{2\,\omega}{a\,\theta} \cot(\omega\, t)$*. Hence for* $t \in (0, t_1]$ *the Mach number is strictly decreasing to* $M(t_1) = 0$*. Since for sufficiently small* t *the state is supersonic and becomes subsonic after finite time, the flow is transsonic on the time-interval* $(0, t_1]$*.*

Remark 4. *Note that in the situation of Theorem 2 the system (1) has a classical solution for all* $t > 0$*. Figure 1 shows the graphs of* $q(t, x)$ *and* $\rho(t, x)$ *for the solution in Theorem 2 for certain parameter values.*

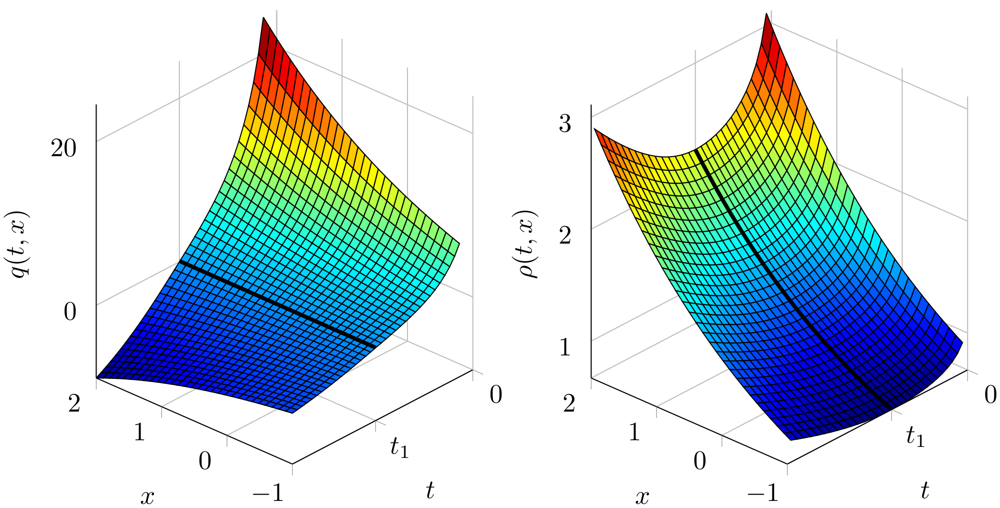

Figure 1: Graphs of $q(t, x)$ and $\rho(t, x)$ for the solution in Theorem 2 for $\theta = 4$, $\beta = 0.4$, $a = 10$, $g = 9.81$, $z^e = -0.005$. Note the unusual orientation of the time-axis: Time is increasing from the back to the front.

Remark 5. *For practical applications subsonic flow is interesting since the case of subsonic flow where the absolute value of the velocity of the gas is strictly less than the sound speed in the gas is the case that is relevant for gas transportation networks. The reason is that if the velocity of the gas in the pipelines is too large, vibrations of the pipes can develop and cause noise pollution. Moreover excessive piping vibration can damage the system. A detailed study of fluid-induced vibration of natural gas pipelines is given in [20].*

The proofs of Theorem 1 and Theorem 2 are presented at the end of this section. The proof of Theorem 1 is based upon the Lemma 1 below, where a product solution of (1) is given explicitly.

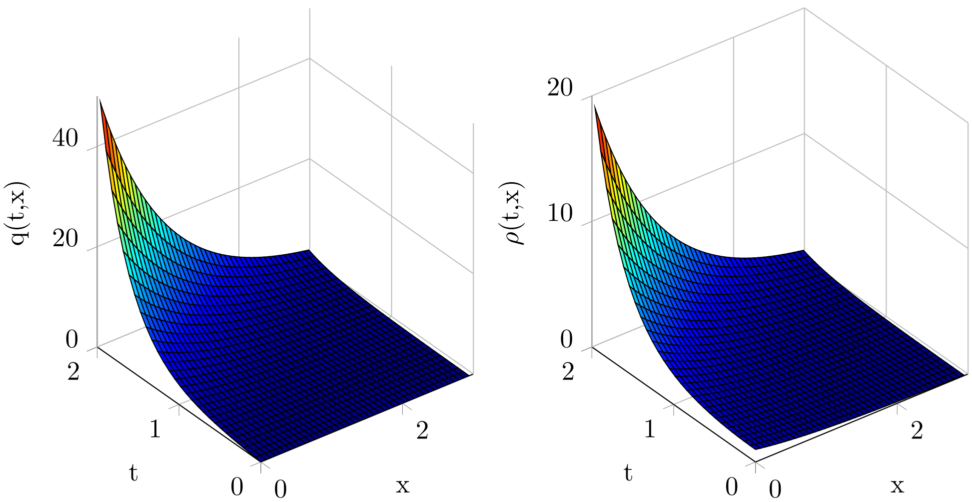

Figure 2: The graphs of $q(t,\, x)$ and $\rho(t,\, x)$ for the solution in Lemma 1 for $\theta = 0.7$, $\beta = -1$, $a = 1$, $z^e = 0$.

Lemma 1. *Let a real number $\beta < 0$ be given such that the inequality $g\, z^e \leq a^2\, |\beta|$ holds. Define the number*

$$\omega = \frac{\sqrt{\theta}}{\sqrt{2}} \sqrt{|a^2\, \beta + g\, z^e|} \tag{12}$$

and for $t \geq 0$ and $x \in (-\infty,\, \infty)$ let

$$q(t,\, x) \quad = \quad \frac{2\,\omega}{\theta}\, \tanh(\omega\, t)\, \cosh(\omega\, t)^{\frac{2\,|\beta|}{\theta}}\, \exp(\beta\, x), \tag{13}$$

$$\rho(t,\, x) \quad = \quad \cosh(\omega\, t)^{\frac{2\,|\beta|}{\theta}}\, \exp(\beta\, x). \tag{14}$$

Then $(\rho,\, q)$ is a classical solution of (1) on the time-interval $[0, \infty)$. The corresponding Mach number is given by

$$M(t) = \frac{\sqrt{2}}{\sqrt{\theta}} \sqrt{|\beta| - \frac{g\, z^e}{a^2}}\, \tanh(\omega\, t)\,. \tag{15}$$

Hence for $t \geq 0$ the Mach number is strictly increasing, remains bounded and approaches a constant for $t \to \infty$. In fact, we have

$$\lim_{t \to \infty} M(t) = \frac{\sqrt{2}}{\sqrt{\theta}} \sqrt{|\beta| - \frac{g\, z^e}{a^2}}.$$

Note that $q(0,\, x) = 0$ and for all $t \geq 0$ we have $q(t, x) \geq 0$. If

$$|\beta| < \frac{\theta}{2} + \frac{g\, z^e}{a^2} \tag{16}$$

then for all $t \geq 0$, the solution is subsonic.

If $|\beta| > \frac{\theta}{2} + \frac{g\, z^e}{a^2}$, the flow converges with time to a supersonic state. Since the initial state is subsonic, this means that the flow is transsonic.

The solution from (13), (14) is the unique solution of the Cauchy problem for (1) with the initial conditions

$$q(0,\, x) = 0, \quad \rho(0,\, x) = \exp(\beta\, x). \tag{17}$$

Figure 2 shows the graphs of $q(t,\, x)$ and $\rho(t,\, x)$ for the solution in Lemma 1 for $\theta = 0.7$, $\beta = -1$, $a = 1$ and $z^e = 0$.

Proof of Lemma 1. We have

$$q_x(t,\, x) = \beta\, \frac{2\,\omega}{\theta}\, \tanh(\omega\, t)\, \cosh(\omega\, t)^{\frac{2\,|\beta|}{\theta}}\, \exp(\beta\, x)$$

and

$$\rho_t(t,\, x) = \omega \frac{2\,|\beta|}{\theta} \cosh(\omega\, t)^{\frac{2\,|\beta|}{\theta} - 1} \sinh(\omega\, t)\, \exp(\beta\, x).$$

Since $\beta < 0$ this implies $\rho_t + q_x = 0$, hence the first equation in (1) holds. Now we look at the second equation in (1). We have

$$q_t(t,\, x) \quad = \quad \frac{2\,\omega^2}{\theta} \frac{1 + \frac{2\,|\beta|}{\theta}\, \sinh^2(\omega\, t)}{\cosh^2(\omega\, t)}\, \cosh(\omega\, t)^{\frac{2\,|\beta|}{\theta}}\, \exp(\beta\, x).$$

Moreover, we have

$$\frac{(q(t,\, x))^2}{\rho(t,\, x)} + a^2 \rho(t,\, x) = \left[a^2 + \left(\frac{2\,\omega}{\theta}\right)^2 \tanh^2(\omega\, t)\right] \cosh(\omega\, t)^{\frac{2\,|\beta|}{\theta}}\, \exp(\beta\, x).$$

This yields the partial derivative with respect to x

$$\left(\frac{q^2}{\rho} + a^2 \rho\right)_x = \beta \left[a^2 + \left(\frac{2\,\omega}{\theta}\right)^2 \tanh^2(\omega\, t)\right] \cosh(\omega\, t)^{\frac{2\,|\beta|}{\theta}}\, \exp(\beta\, x).$$

Hence we obtain for the left-hand side of the second equation in (1)

$$q_t + \left(\frac{q^2}{\rho} + a^2 \rho\right)_x$$

$$= \left[a^2 \beta + \frac{2\,\omega^2}{\theta} \frac{1}{\cosh^2(\omega\, t)}\right] \cosh(\omega\, t)^{\frac{2\,|\beta|}{\theta}}\, \exp(\beta\, x).$$

On the other hand for all $t \geq 0$ we have $q \geq 0$ and hence

$$-\frac{1}{2}\, \theta\, \frac{q\, |q|}{\rho} - \rho\, g\, z^e$$

$$= -\frac{1}{2} \theta \left(\frac{2\,\omega}{\theta}\right)^2 \tanh^2(\omega\, t)\, \cosh(\omega\, t)^{\frac{2\,|\beta|}{\theta}}\, \exp(\beta\, x)$$

$$
\begin{aligned}
& -g\, z^e\, \cosh(\omega\, t)^{\frac{2\,|\beta|}{\theta}}\, \exp(\beta\, x) \\
= & \left[-\frac{2\,\omega^2}{\theta} \tanh^2(\omega\, t) - g\, z^e \right] \cosh(\omega\, t)^{\frac{2\,|\beta|}{\theta}}\, \exp(\beta\, x).
\end{aligned}
$$

Due to the properties of the hyperbolic functions and the definition of ω, we have

$$
\begin{aligned}
a^2\beta + \frac{2\,\omega^2}{\theta} \frac{1}{\cosh^2(\omega\, t)} &= a^2\beta + \frac{2\,\omega^2}{\theta}\left(1 - \tanh^2(\omega\, t)\right) \\
&= \frac{a^2\,\theta\,\beta + 2\,\omega^2}{\theta} - \frac{2\,\omega^2}{\theta} \tanh^2(\omega\, t) \\
&= -\frac{2\,\omega^2}{\theta} \tanh^2(\omega\, t) - g\, z^e.
\end{aligned}
$$

Hence the second equation in (1) is also satisfied. The uniqueness of the solution follows from the uniqueness of the semi–global classical solutions, see [12]. Thus we have proved Lemma 1.

Proof of Theorem 1. Let $K \in \{1, 2, 3, ...\}$ be given. Choose a natural number $N \geq \max\{K, \frac{g\, z^e}{a^2}\}$. Define $\beta = -N < 0$ and $\omega = \frac{\sqrt{\theta}}{\sqrt{2}} \sqrt{a^2\, N - g\, z^e}$. Then the solution $(q,\, \rho)$ of (1) presented in Lemma 1 is well–defined. We have $|\rho_x(0,\, 0)| = |\beta| \geq K$. Define $\rho_0(x) = \rho(0,\, x)$ and $q_0(x) = q(0, x)$. Then $\rho_0(0) = 1$ and (5) holds. Since $\frac{\rho_0'(0)}{\rho_0(0)} = \beta$, the inequality (6) also holds. In Lemma 1 it has been shown that the initial state $(\rho_0(x),\, q_0(x))$ generates a global classical solution of (1). Hence the first statement of Theorem 1 is proved. The second statement about the Mach number follows from (15).

In Lemma 1 we have presented a global classical solution with hyperbolic functions that started with a zero flow rate. In Lemma 2 we present a classical solution with trigonometric functions that is defined on a finite time interval $(0, t_1]$ only until the flow rate reaches zero. For larger times $t > t_1$ the solution can be continued as a classical solution that is represented by hyperbolic functions. In Lemma 2 in contrast to Lemma 1, we have $\beta > 0$.

Lemma 2. *Let $\theta > 0$ and $\beta > 0$ be given such that $g\,|z^e| \leq a^2\,\beta$. Define ω as in (12) and for $t \in (0,\, \frac{\pi}{2\,\omega}]$ and $x \in (-\infty,\, \infty)$ let*

$$
q(t,\, x) = \frac{2\,\omega}{\theta} \cot(\omega\, t)\, \sin(\omega\, t)^{-\frac{2\,|\beta|}{\theta}}\, \exp(\beta\, x), \tag{18}
$$

$$
\rho(t,\, x) = \sin(\omega\, t)^{-\frac{2\,|\beta|}{\theta}}\, \exp(\beta\, x). \tag{19}
$$

Define $t_1 = \frac{\pi}{2}\frac{1}{\omega}$. Then $(\rho,\, q)$ is a classical solution of (1) on the time-interval $(0, t_1]$. It is the unique solution that satisfies the conditions $q(t_1,\, x) = 0$ and $\rho(t_1,\, x) = \exp(\beta\, x)$. For $t \in (0,\, t_1]$, the Mach number is given by

$$
M(t) = \frac{q}{a\,\rho} = \frac{2\,\omega}{a\,\theta} \cot(\omega\, t)\,.
$$

Hence for $t \in (0, t_1]$ the Mach number is strictly decreasing to $M(t_1) = 0$.

Since for sufficiently small t the state is supersonic, the flow is transsonic on the time-interval $(0, t_1]$.

Proof of Lemma 2. We have

$$q_x(t,\, x) = \beta\, \frac{2\,\omega}{\theta}\, \cot(\omega\, t)\, \sin(\omega\, t)^{-\frac{2\,|\beta|}{\theta}}\, \exp(\beta\, x)$$

and

$$\rho_t(t,\, x) = -\omega \frac{2\,|\beta|}{\theta} \sin(\omega\, t)^{-\frac{2\,|\beta|}{\theta}-1} \cos(\omega\, t)\, \exp(\beta\, x).$$

Since $\beta > 0$ this implies $\rho_t + q_x = 0$, hence the first equation in (1) holds. Now we look at the second equation in (1). We have

$$q_t(t,\, x) \quad = \quad \frac{2\,\omega^2}{\theta}\, \frac{-1 - \frac{2\,|\beta|}{\theta}\cos^2(\omega\, t)}{\sin^2(\omega\, t)}\, \sin(\omega\, t)^{\frac{-2\,|\beta|}{\theta}}\, \exp(\beta\, x).$$

Moreover, we have

$$\frac{(q(t,\, x))^2}{\rho(t,\, x)} + a^2 \rho(t,\, x) = \left[a^2 + \left(\frac{2\,\omega}{\theta}\right)^2 \cot^2(\omega\, t)\right] \sin(\omega\, t)^{-\frac{2\,|\beta|}{\theta}}\, \exp(\beta\, x).$$

This yields

$$\left(\frac{q^2}{\rho} + a^2 \rho\right)_x = \beta \left[a^2 + \left(\frac{2\,\omega}{\theta}\right)^2 \cot^2(\omega\, t)\right] \sin(\omega\, t)^{\frac{-2\,|\beta|}{\theta}}\, \exp(\beta\, x).$$

Hence we obtain for the left-hand side of the second equation in (1)

$$q_t + \left(\frac{q^2}{\rho} + a^2 \rho\right)_x$$

$$= \left[a^2 \beta - \frac{2\,\omega^2}{\theta}\frac{1}{\sin^2(\omega\, t)} + \left(-\frac{2\,\omega^2}{\theta}\frac{2\,|\beta|}{\theta} + \beta\left(\frac{2\,\omega}{\theta}\right)^2\right)\cot^2(\omega\, t)\right] \sin(\omega\, t)^{\frac{-2\,|\beta|}{\theta}}\, \exp(\beta\, x)$$

$$= \left[a^2 \beta - \frac{2\,\omega^2}{\theta}\frac{1}{\sin^2(\omega\, t)}\right] \sin(\omega\, t)^{-\frac{2\,|\beta|}{\theta}}\, \exp(\beta\, x).$$

On the other hand for all $t \in (0,\, t_1]$ we have $q \geq 0$ and hence

$$-\frac{1}{2}\,\theta\, \frac{q\,|q|}{\rho} - \rho\, g\, z^e$$

$$= -\frac{1}{2}\theta \left(\frac{2\,\omega}{\theta}\right)^2 \cot^2(\omega\, t)\, \sin(\omega\, t)^{-\frac{2\,|\beta|}{\theta}}\, \exp(\beta\, x)$$

$$-g\, z^e\, \sin(\omega\, t)^{-\frac{2\,|\beta|}{\theta}}\, \exp(\beta\, x)$$

$$= \left[-\frac{2\,\omega^2}{\theta}\cot^2(\omega\, t) - g\, z^e\right] \sin(\omega\, t)^{-\frac{2\,|\beta|}{\theta}}\, \exp(\beta\, x).$$

Due to the properties of the trigonometric functions and the definition of ω, we have

$$\begin{aligned} a^2\beta - \frac{2\,\omega^2}{\theta}\frac{1}{\sin^2(\omega\,t)} &= a^2\beta - \frac{2\,\omega^2}{\theta}\left(1+\cot^2(\omega\,t)\right) \\ &= \frac{a^2\,\theta\,\beta - 2\,\omega^2}{\theta} - \frac{2\,\omega^2}{\theta}\,\cot^2(\omega\,t) \\ &= -\frac{2\,\omega^2}{\theta}\,\cot^2(\omega\,t) - g\,z^e. \end{aligned}$$

Hence the second equation in (1) is also satisfied and the assertion follows. Thus we have proved Lemma 2.

Remark 6. *The solutions that we have presented in this section yield examples for instationary initial states that generate solutions that exist without blow-up for all times $t > 0$. These initial states are given by positive strictly decreasing exponential profiles for q and ρ with a quotient $\frac{q(t,\,\cdot)}{\rho(t,\,\cdot)}$ that is independent of x and only depends on t.*

Proof of Theorem 2. Now Theorem 2 follows using Lemma 2 for $t \in (0,\, t_1]$. For $t > t_1$, the arguments are similar as in the proof of Lemma 1.

4. A semilinear model

For semilinear models in contrast to the quasilinear models, the generation of shocks from smooth initial states in finite time does not occur. The easiest way to obtain a semilinear model from (1) is to cancel the quadratic term in the left-hand side of the second equation. This yields the semilinear hyperbolic partial differential equation

$$\left\{ \begin{array}{l} \rho_t + q_x = 0, \\ q_t + a^2\rho_x = -\frac{1}{2}\theta\frac{q\,|q|}{\rho} - \rho\,g\,z^e. \end{array} \right. \tag{20}$$

The analytical solutions that we have derived for (1) give us the opportunity to compare them with the states that are generated by (20) if the system is started with the same initial data. Using the Mach number M, we can write the quadratic term that is canceled in (20) as

$$\frac{q^2}{\rho} = a\,M\,q.$$

For the product solutions that we have constructed we have $M_x = 0$, thus

$$\left(\frac{q^2}{\rho}\right)_x = a\,M\,q_x.$$

For subsonic solutions with $q_x = \beta\,q$ this yields the bound

$$\left|\left(\frac{q^2}{\rho}\right)_x\right| \le a\,|\beta|\,|q|.$$

Thus for small values of $|\beta|$, for the solutions considered in Section 3, this indicates that the term that is cancelled when we go from the quasilinear model (1) to the semilinear model (20) is relatively small compared to $|q|$. In order to compare solutions of (1) and (20), we look at product solutions of (20).

4.1. Product solution for the semilinear model with exponential profile in space

In this section we give a derivation of product solutions for the semilinear partial differential equation (20). This allows us to determine the exactly the difference to the solutions of the quasilinear model (1). We have the following result.

Lemma 3. *Let $\theta > 0$ and $\beta < 0$ be given such that*

$$\left(a^2\,\beta^2 + g\,z^e\,\beta\right)\left(1 - \frac{\theta}{2\,\beta}\right) > 0. \tag{21}$$

Define

$$\omega = \sqrt{a^2\,\beta^2 + g\,z^e\,\beta - \frac{\theta}{2}\,(a^2\,\beta + g\,z^e)} \tag{22}$$

and for $t \geq 0$ and $x \in (-\infty,\, \infty)$ let

$$q(t,\, x) \quad = \quad \frac{2\,\omega}{\theta + 2\,|\beta|}\,\tanh{(\omega\, t)}\,\cosh{(\omega\, t)}^{\frac{2\,|\beta|}{\theta+2|\beta|}}\,\exp(\beta\, x), \tag{23}$$

$$\rho(t,\, x) \quad = \quad \cosh{(\omega\, t)}^{\frac{2\,|\beta|}{\theta+2|\beta|}}\,\exp(\beta\, x). \tag{24}$$

Then $(\rho,\, q)$ is a classical solution of (20) on the time-interval $[0, \infty)$. The corresponding Mach number is given by

$$M(t) = \frac{2\,\omega}{a\,(\theta - 2\,\beta)}\,\tanh{(\omega\, t)}\,.$$

Hence for $t \geq 0$ the Mach number is strictly increasing, remains bounded and approaches a constant for $t \to \infty$. In fact, we have

$$\lim_{t\to\infty} M(t) = \frac{2\,\omega}{a\,(\theta - 2\,\beta)}.$$

Note that $q(0,\, x) = 0$ and for all $t \geq 0$ we have $q(t, x) \geq 0$. If

$$2\,\omega < a\,(\theta - 2\,\beta) \tag{25}$$

then for all $t \geq 0$, the solution is subsonic.

If $2\,\omega > a\,(\theta - 2\,\beta)$, the flow converges with time to a supersonic state. Since the initial state is subsonic, this means that the flow is transsonic.

The solution from (23), (24) is the unique solution of the Cauchy problem for (20) with the initial conditions

$$q(0,\, x) = 0, \quad \rho(0,\, x) = \exp(\beta\, x). \tag{26}$$

Proof of Lemma 3. We have

$$q_x(t,\, x) = \beta\, \frac{2\,\omega}{\theta + 2\,|\beta|}\ \tanh(\omega\, t)\, \cosh(\omega\, t)^{\frac{2\,|\beta|}{\theta+2\,|\beta|}}\ \exp(\beta\, x)$$

and

$$\rho_t(t,\, x) = \omega \frac{2\,|\beta|}{\theta + 2\,|\beta|} \cosh(\omega\, t)^{\frac{2\,|\beta|}{\theta+2\,|\beta|}-1} \sinh(\omega\, t)\ \exp(\beta\, x).$$

Since $\beta < 0$ this implies $\rho_t + q_x = 0$, hence the first equation in (20) holds. Now we look at the second equation in (20). We have

$$q_t(t,\, x) \quad = \quad \frac{2\,\omega^2}{\theta + 2\,|\beta|} \frac{1 + \frac{2\,|\beta|}{\theta+2\,|\beta|}\ \sinh^2(\omega\, t)}{\cosh^2(\omega\, t)}\ \cosh(\omega\, t)^{\frac{2\,|\beta|}{\theta+2\,|\beta|}}\ \exp(\beta\, x).$$

Hence we obtain for the left-hand side of the second equation in (20)

$$q_t + a^2 \rho_x$$

$$= \left[a^2\, \beta + \frac{2\,\omega^2}{\theta + 2\,|\beta|} \frac{1}{\cosh^2(\omega\, t)} + \frac{4\,\omega^2\,|\beta|}{(\theta + 2\,|\beta|)^2}\ \tanh^2(\omega\, t)\right] \cosh(\omega\, t)^{\frac{2\,|\beta|}{\theta+2\,|\beta|}}\ \exp(\beta\, x).$$

On the other hand for all $t \geq 0$ we have $q \geq 0$ and hence

$$-\frac{1}{2}\,\theta\, \frac{q\ |q|}{\rho} - \rho\, g\, z^e$$

$$= -\frac{1}{2}\,\theta \left(\frac{2\,\omega}{\theta + 2\,|\beta|}\right)^2 \tanh^2(\omega\, t)\, \cosh(\omega\, t)^{\frac{2\,|\beta|}{\theta+2\,|\beta|}}\ \exp(\beta\, x)$$

$$-g\, z^e\ \cosh(\omega\, t)^{\frac{2\,|\beta|}{\theta+2\,|\beta|}}\ \exp(\beta\, x)$$

$$= \left[-\theta\, \frac{2\,\omega^2}{(\theta + 2\,|\beta|)^2}\, \tanh^2(\omega\, t) - g\, z^e\right] \cosh(\omega\, t)^{\frac{2\,|\beta|}{\theta+2\,|\beta|}}\ \exp(\beta\, x).$$

Due to the properties of the hyperbolic functions and the definition of ω, we have

$$\begin{aligned}
& a^2\beta + \frac{2\,\omega^2}{\theta + 2\,|\beta|} \frac{1}{\cosh^2(\omega\, t)} + \frac{4\,\omega^2\,|\beta|}{(\theta + 2\,|\beta|)^2}\ \tanh^2(\omega\, t) \\
= \quad & a^2\beta + \frac{2\,\omega^2}{\theta + 2\,|\beta|} \left(1 - \tanh^2(\omega\, t)\right) + \frac{4\,\omega^2\,|\beta|}{(\theta + 2\,|\beta|)^2}\ \tanh^2(\omega\, t) \\
= \quad & \frac{a^2\,\theta\,\beta + 2\,a^2\,\beta\,|\beta| + 2\,\omega^2}{\theta + 2\,|\beta|} - \frac{2\,\omega^2}{\theta + 2\,|\beta|}\ \tanh^2(\omega\, t) + \frac{4\,\omega^2\,|\beta|}{(\theta + 2\,|\beta|)^2}\ \tanh^2(\omega\, t) \\
= \quad & -\frac{2\,\omega^2\,\theta}{(\theta + 2\,|\beta|)^2}\ \tanh^2(\omega\, t) - g\, z^e.
\end{aligned}$$

Hence the second equation in (20) is also satisfied. Again the uniqueness of the solution of the Cauchy–problem follows from the uniqueness of the semi–global classical solutions, see [12]. Thus we have proved Lemma 3.

Remark 7. *Lemma 3 illustrates that the form of the solutions of the semilinear model (20) is similar as in Lemma 1. In fact, only the exponents in the definitions of q and ρ and the denominators in (13) and (23) are different. Note however, that the value of the parameter ω that appears in the solutions is different. Let ω_1 denote ω as defined in (12) and ω_2 denote ω as defined in (22). If $g\, z^2 \leq a^2\, |\beta|$ and (21) holds, we have*

$$\omega_2^2 - \omega_1^2 = a^2\, \beta^2 + g\, z^e\, \beta.$$

Let (q_1, ρ_1) denote the solution of the Cauchy problem with the initial condition (26) and the quasilinear model (1) and let (q_2, ρ_2) denote the solution of the semilinear initial value problem (26), (20). Due to our results, we can determine the difference between these solutions. In fact, we have

$$(q_1 - q_2)(t,\, x) = \left[\frac{2\,\omega_1}{\theta}\, \tanh(\omega_1\, t)\, \cosh(\omega_1\, t)^{\frac{2\,|\beta|}{\theta}}\right.$$

$$\left. - \frac{2\,\omega_2}{\theta + 2\,|\beta|}\, \tanh(\omega_2\, t)\, \cosh(\omega_2\, t)^{\frac{2\,|\beta|}{\theta+2|\beta|}}\right] \exp(\beta\, x),$$

$$(\rho_1 - \rho_2)(t,\, x) = \left[\cosh(\omega_1\, t)^{\frac{2\,|\beta|}{\theta}} - \cosh(\omega_2\, t)^{\frac{2\,|\beta|}{\theta+2|\beta|}}\right] \exp(\beta\, x).$$

Remark 8. *In [3], the following model (FD1) for the case of friction dominated flow is discussed, and it is stated that models* of this type are often used and are well known in the gas pipeline context *(see for example [11]):*

$$\begin{cases} \rho_t + q_x = 0, \\ a^2 \rho_x = -\frac{1}{2}\theta \frac{q\,|q|}{\rho} - \rho\, g\, z^e. \end{cases} \tag{27}$$

For $\beta \neq 0$, the corresponding product solutions are of the form

$$\rho(t,\, x) = \exp(\mu\, t)\, \exp(\beta\, x), \quad q(t,\, x) = -\frac{\mu}{\beta}\, \exp(\mu\, t)\, \exp(\beta\, x) \tag{28}$$

where the real number μ is chosen such that

$$\theta\, \mu\, |\mu| = -2\, a^2 \beta^2\, |\beta| - 2\, g\, z^e \beta |\beta|. \tag{29}$$

So we see that in this case our product solutions reduce to exponential travelling waves (as functions of $(\mu\, t + \beta x)$) and thus the type of the solutions changes completely due to the change in the model.

An error analysis for the Euler equations in purely algebraic form, also called the Weymouth equations, is given in [16]. This is a stationary model, so the solutions are of a different type than the instationary states that we have considered in this paper. Let us emphasize again that the instationary states are important in practice. Already in [18], a 24–hour cycle illustrates the changes in consumer demand within a day.

5. Conclusion

We have presented product solutions of the one–dimensional isothermal Euler equations for ideal gas with source term. Our solutions can be used to test numerical methods. Moreover, they provide some analytical insight into the system. We have presented a global classical solution where the direction of flow is reversed. We have also shown that for sufficiently large friction parameters, subsonic global classical solutions with arbitrarily large logarithmic derivatives exist. In addition, we have discussed different monotonicity types that appear during the flow.

The analytical solutions that are presented in this paper are important for the understanding of the system since they show that also for initial states with arbitrarily large derivatives, global classical solutions can exist, whereas in general existence results, usually a smallness assumption for the initial states appears. The solutions are also important since they allow to study the error that is generated if the isothermal Euler equations are replaced by simpler models, for example semilinear hyperbolic systems.

Pipelines often form a complex networked system. Therefore it would be useful to have transient solutions that are defined on network graphs, where the flow through the nodes is governed by algebraic node conditions (see [7]). This is a topic for future research. Another open question is whether similar solutions exist for other models of gas, for example with a non–constant compressibility factor as in [10] or with a different equation of state for isentropic gas.

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