ON THE EXISTENCE, UNIQUENESS AND EXACT CONTROLLABILITY OF LIPSCHITZ SOLUTIONS OF INITIAL BOUNDARY VALUE PROBLEMS FOR GAS NETWORKS WITH NONCONSTANT COMPRESSIBILITY FACTOR

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Abstract. The flow of gas through networks of pipes can be modeled by the isothermal Euler equations and algebraic node conditions that model the flow through the vertices of the network graph. We prove the well-posedness of the system for gas with nonconstant compressibility factor that is given by an affine linear function.

We consider initial data and control functions that are Lipschitz continuous and compatible with the node and boundary conditions. We show the existence of semi–global Lipschitz continuous solutions of the initial boundary value problem. The construction of the solution is based upon a fixed point iteration along the characteristic curves. This allows us to study the exact controllability of the system for tree–shaped networks. For a steering time that is sufficiently large and stationary states that are sufficiently small the system is locally exactly controllable by boundary controls at all boundary nodes.

The solutions of the intial boundary value problem on arbitrary networks satisfy a maximum principle in terms of the Riemann invariants that states that the maximum of the absolute values is attained for the initial or the boundary data.

Key words. isothermal Euler equations, z-factor, compressibility factor, real gas, network, node conditions, gas transportation network, Lipschitz continuity, maximum principle, exact controllability, tree–shaped graph.

AMS subject classifications. 35L04, 49K20, 90C46

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1. Introduction.

We prove the existence of a unique solution of an initial boundary value problem with the isothermal Euler equations for real gas with an affine linear compressibility factor on a network of pipes. For this quasilinear system, we show the existence of a semi-global solution, that is we provide conditions that guarantee that the solution exists on a given finite time interval $[0, T]$ and satisfies an a-priori estimate. We assume that the initial and the boundary data are Lipschitz continuous with sufficiently small Lipschitz constant. The initial data also have to be compatible with the node conditions in the network, that require the conservation of mass and the continuity of the pressure at the nodes. Moreover, the initial and the boundary data have to be compatible as continuous functions and the friction term has to be sufficiently small.

We consider solutions that satisfy the partial differential equation in the sense of the characteristic curves. The solution is continuous and Lipschitz continuous with respect to the space variable.

The solutions are constructed by a fixed point Picard-iteration along the characteristic curves. This approach is similar to the construction of classical solutions, see for example [21]. However, one novelty of our contribution is that in contrast to the classical solutions, our solutions need not be differentiable, but they still generate a

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family of non–intersecting characteristic curves without rarefaction fans.

The existence of classical solutions and feedback stabilization for the flow of ideal gas in networks is studied in [8]. A generalization to a more general class of networks is given in [12]. Problems of optimal control in networks of pipes and canals are considered in [4], where the existence of optimal control is proved for weak entropy solutions of the state equations with L^1 controls with bounded variation. The results in [4] also apply to the real gas model that we consider in this paper. However, in this paper we consider the initial-boundary value problem with Lipschitz continuous boundary controls and initial data that generate more regular Lipschitz continuous states. Lipschitz continuous solutions for the Lighthill-Whitham-Richards model (LWR) for traffic flow have been studied in [11].

The exact controllability of networked systems governed by quasilinear partial differential equations on trees has been studied before in the framework of classical solutions for example in [20], where the Saint–Venant system with zero source terms is considered, and in [7], where quasilinear wave equations are considered. The exact controllability of generic trees of elastic homogeneous strings is treated in [19]. The global boundary controllability of the Saint-Venant system for sloped canals with friction is studied in [16]. An overview on recent results on flows on networks is given in [3].

In this paper, we show the local exact controllability to stationary states on trees for Lipschitz continuous solutions of our system. Moreover, we show that for networks corresponding to arbitrary graphs the solutions of the intial boundary value problem satisfy a maximum principle in terms of the Riemann invariants that states that the maximum of the absolute values of the Riemann invariants is attained for the initial state or the values at the boundary nodes.

This paper has the following structure. In Section 2 we introduce the isothermal Euler equations and the affine linear compressibility factor. In Section 3, the partial differential equation is transformed to the diagonal form. and the Riemann invariants are presented. In Section 4, the node conditions for the flow through the junctions in the network are given. In Section 5, the short–time existence of solutions of solutions that satisfy an a—priori estimate is shown in Theorem 2. Due to the a priori–estimate, this allows to prove the existence of semi–global solutions. The corresponding result is given in Theorem 4. In Section 5.1, we present a maximum principle for the solution. In Section 6, the local exact controllability to stationary states is proved for networks that are given by trees.

2. The isothermal Euler equations. Let a finite directed graph $G = (V, E)$ of a pipeline network be given. Here V denotes the set of vertices and E denotes the set of edges. Each edge $e \in E$ corresponds to an interval $[0, L^e]$ that represents a pipe of length L^e. Let $D^e > 0$ denote the diameter of the pipe, $\lambda^e_{fric}(x) \geqslant 0$ the space-dependent Lipschitz continuous friction coefficient and $\varphi^e(x) \in (-\infty, \infty)$ the space-dependent Lipschitz continuous slope. Define $s^e_{lope}(x) = \sin(\varphi^e(x))$ and $\theta^e(x) = \frac{\lambda^e_{fric}(x)}{D^e}$. Let g denote the gravitational constant. Let ρ^e denote the gas density, p^e the pressure and q^e the mass flow rate. Let $\alpha^e \in (-0.5, 0)$ be given and define the compressibility factor as

$$z^e(p^e) = 1 + \alpha^e\, p^e. \tag{1}$$

Equation (1) is also stated in [23] as the model of the American Gas Association (AGA). In [1] it is stated that it is sufficiently accurate within the network operating range. We assume that

$$p^e = R_s^e\, T^e\, z^e(p^e)\, \rho^e, \tag{2}$$

where R_s^e is the gas constant and T^e is the temperature. We study the isothermal Euler equations

$$\begin{cases} \rho_t^e + q_x^e = 0, \\ q_t^e + \left(p^e + \frac{(q^e)^2}{\rho^e}\right)_x = -\frac{1}{2}\theta^e \frac{q^e\,|q^e|}{\rho^e} - \rho^e\, g\, s_{lope}^e \end{cases} \tag{3}$$

that govern the flow through a single pipe. In our analysis also the velocity

$$v^e = \frac{q^e}{\rho^e}$$

and the sound speed c^e that is given by

$$\left(\frac{1}{c^e}\right)^2 = \frac{\partial \rho^e}{\partial p^e} = \frac{1}{R_s^e\, T^e\, (1 + \alpha^e\, p^e)^2} \tag{4}$$

appear. For the Mach number M^e this yields

$$M^e = \frac{v^e}{c^e} = \sqrt{R_s^e\, T^e}\, \frac{q^e}{p^e}. \tag{5}$$

Stationary states for the case of an ideal gas, that is for $\alpha^e = 0$, have been considered in [14]. In this paper we focus on the case of real gas where $\alpha^e < 0$. Stationary states for this case have been studied in [18]. From the first equation in (3), (2) yields

$$p_t^e + R_s^e\, T^e (1 + \alpha^e\, p^e)^2\, q_x^e = 0. \tag{6}$$

From the second equation in (3), we get

$$\begin{aligned} & q_t^e + \left(1 - R_s^e\, T^e \frac{(q^e)^2}{(p^e)^2}\right) p_x + 2 R_s^e\, T^e \left(\frac{q^e}{p^e} + \alpha^e\, q^e\right) q_x^e \\ = \; & -\frac{1}{2}\theta^e R_s^e\, T^e (1 + \alpha^e\, p^e)\, \frac{q^e\,|q^e|}{p^e} - \frac{1}{R_s^e\, T^e}\, \frac{p^e}{(1+\alpha^e\, p^e)}\, g\, s_{lope}^e. \end{aligned} \tag{7}$$

We consider the case of *subsonic* flow where the absolute value of the velocity of the gas is strictly less than the sound speed in the gas, that is for the Mach number we have

$$|M^e| < 1.$$

This is the case that is relevant for gas transportation networks, because if the velocity of the gas in the pipelines is too large, vibrations of the pipes can develop and cause noise pollution. Moreover excessive piping vibration can damage the system. Therefore, there are upper bounds for the velocity of the gas in the operation of gas pipelines. A detailed study of fluid-induced vibration of natural gas pipelines is given in [24].

3. The equations in Riemann invariants. Now we transform the system in $(p^e,\, q^e)$ in diagonal form to obtain the Riemann invariants. We write (6)-(7) in the form

$$\begin{pmatrix} p^e \\ q^e \end{pmatrix}_t + A^e(p^e, q^e) \begin{pmatrix} p^e \\ q^e \end{pmatrix}_x = F^e(p^e, q^e) \tag{8}$$

with the matrix

$$A^e(p^e, q^e) = \begin{pmatrix} 0 & R_s^e\, T^e (1 + \alpha^e\, p^e)^2 \\ 1 - R_s^e\, T^e \frac{(q^e)^2}{(p^e)^2} & 2 R_s^e\, T^e \left(\frac{q^e}{p^e} + \alpha^e\, q^e \right) \end{pmatrix}$$
$$= \begin{pmatrix} 0 & (c^e)^2 \\ 1 - (M^e)^2 & 2 (c^e)(M^e) \end{pmatrix}.$$

The matrix A^e has the left-hand side eigenvectors

$$l_-^e = \frac{1}{p^e}\,(1 + M^e, -c^e)$$

that correspond to the negative eigenvalue

$$\lambda_-^e = c^e\,(M^e - 1) = v^e - c^e < 0$$

(that is $l_-^e\, A^e = \lambda_-^e l_-^e$) and the left-hand side eigenvectors

$$l_+^e = \frac{1}{p^e}\,(1 - M^e, c^e)$$

that correspond to the positive eigenvalue

$$\lambda_+^e = c^e\,(1 + M^e) = v^e + c^e > 0$$

(that is $l_+^e\, A^e = \lambda_+^e l_+^e$). We have

$$l_-^e = (\partial_{p^e} R_-^e,\, \partial_{q^e} R_-^e),\ l_+^e = (\partial_{p^e} R_+^e,\, \partial_{q^e} R_+^e) \tag{9}$$

where R_-^e, R_+^e are the Riemann invariants

$$R_-^e(p^e,\, q^e) = \ln(p^e) - \sqrt{R_s^e\, T^e}\, \frac{q^e}{p^e}\,(1 + \alpha^e\, p^e)\,,$$

$$R_+^e(p^e,\, q^e) = \ln(p^e) + \sqrt{R_s^e\, T^e}\, \frac{q^e}{p^e}\,(1 + \alpha^e\, p^e)\,.$$

We have

$$\frac{R_+^e - R_-^e}{2} = \frac{1}{\sqrt{R_s^e\, T^e}}\, c^e\, M^e \tag{10}$$

and

$$\frac{R_+^e + R_-^e}{2} = \ln(p^e). \tag{11}$$

Hence

$$p^e = \exp\left(\frac{R_+^e + R_-^e}{2} \right),$$

and

$$q^e = \frac{1}{\sqrt{R_s^e\, T^e}}\, \frac{p^e}{(1 + \alpha^e\, p^e)}\, \left(\frac{R_+^e - R_-^e}{2} \right).$$

Thus we can express the eigenvalues λ_-^e, λ_+^e in terms of the Riemann invariants as

$$\lambda_-^e = \sqrt{R_s^e T^e}\left[\frac{R_+^e - R_-^e}{2} - 1 - \alpha^e \exp\left(\frac{R_+^e + R_-^e}{2}\right)\right], \tag{12}$$

$$\lambda_+^e = \sqrt{R_s^e T^e}\left[\frac{R_+^e - R_-^e}{2} + 1 + \alpha^e \exp\left(\frac{R_+^e + R_-^e}{2}\right)\right]. \tag{13}$$

For our analysis, it is important that the eigenvalues λ_+^e, λ_-^e are Lipschitz continuous as functions of the Riemann invariants R_+^e, R_-^e on bounded sets. This fact is stated in the following Lemma, that contains an explicit Lipschitz constant and lower and upper bounds for the eigenvalues as functions of the Riemann invariants on the bounded square $M_1(u_{\max}) = [-u_{\max}, u_{\max}]^2$.

Lemma 1. *For a given number $u_{\max} > 0$ define the set*

$$M_1(u_{\max}) = \{(R_+^e, R_-^e) : |R_+^e| \leqslant u_{\max},\ |R_-^e| \leqslant u_{\max}\} = [-u_{\max}, u_{\max}]^2 \tag{14}$$

and the number

$$\Lambda^e(u_{\max}) = \sqrt{R_s^e T^e}\,[1 + |\alpha^e| \exp(u_{\max}) + u_{\max}]\,. \tag{15}$$

Then for $(R_+^e, R_-^e) \in M_1(u_{\max})$ and λ_-^e, λ_+^e as defined in (12), (13) we have

$$\max\{|\lambda_+^e|,\, |\lambda_-^e|\} \leqslant \Lambda^e(u_{\max}).$$

The eigenvalues λ_+^e, λ_-^e are Lipschitz continuous functions on the set $M_1(u_{\max})$ in the sense that for all (R_+^e, R_-^e), $(S_+^e, S_-^e) \in M_1(u_{\max})$ they satisfy the Lipschitz inequality

$$|\lambda_\pm^e(R_+^e, R_-^e) - \lambda_\pm^e(S_+^e, S_-^e)| \leqslant \frac{\sqrt{R^e\, T^e}}{2}\,(1 + |\alpha^e| \exp(u_{\max}))\,\left[|R_+^e - S_+^e| + |R_-^e - S_-^e|\right]$$

$$\leqslant \frac{\Lambda^e(u_{\max})}{2}\,\left[|R_+^e - S_+^e| + |R_-^e - S_-^e|\right]. \tag{16}$$

Define the number

$$\underline{\Lambda}^e(u_{\max}) = \sqrt{R_s^e T^e}\,(1 - |\alpha^e| \exp(u_{\max}) - u_{\max})\,. \tag{17}$$

Then we have the inequalities

$$\lambda_+^e(R_+^e, R_-^e) \geqslant \underline{\Lambda}^e(u_{\max}),\ \lambda_-^e(R_+^e, R_-^e) \leqslant -\underline{\Lambda}^e(u_{\max}).$$

Hence if

$$u_{\max} < 1 - |\alpha^e|\ \exp(u_{\max}), \tag{18}$$

on the set $M_1(u_{\max})$ we have $\lambda_+^e > 0$ and $\lambda_-^e < 0$, that is the flow is subsonic.

Lemma 1 follows directly from (12), (13) so we omit the details of the proof here.

Define the matrix $\hat{L}^e$ that has the eigenvectors l_+^e, l_-^e as rows, that is

$$\hat{L}^e = \begin{pmatrix} l_+^e \\ l_-^e \end{pmatrix}.$$

In order to write the pde-system (6)-(7) in terms of $(R_+^e,\, R_-^e)$ we multiply (8) from the left-hand side by the matrix $\hat{L}^e$ and get

$$\hat{L}^e \begin{pmatrix} p^e \\ q^e \end{pmatrix}_t + \operatorname{diag}(\lambda_+^e, \lambda_-^e)\, \hat{L}^e \begin{pmatrix} p^e \\ q^e \end{pmatrix}_x = \hat{L}^e\, F^e(p^e, q^e). \tag{19}$$

Due to (9) this yields

$$\begin{pmatrix} R_+^e \\ R_-^e \end{pmatrix}_t + \operatorname{diag}(\lambda_+^e, \lambda_-^e) \begin{pmatrix} R_+^e \\ R_-^e \end{pmatrix}_x = \hat{L}^e\, F^e(p^e, q^e). \tag{20}$$

Now we also express the source term in the Riemann invariants. We have

$$\begin{aligned} \hat{L}^e\, F^e(p^e, q^e) &= \left[\frac{1}{2}\, \theta^e (R_s^e\, T^e)^{3/2}\, (1 + \alpha^e\, p^e)^2\, \frac{q^e\, |q^e|}{(p^e)^2} + \frac{1}{\sqrt{R_s^e\, T^e}}\, g\, s_{lope}^e \right] \begin{pmatrix} -1 \\ 1 \end{pmatrix} \\ &= \left[\frac{1}{2}\, \theta^e \frac{\operatorname{sign}(q^e)}{\sqrt{R_s^e\, T^e}}\, (c^e)^2\, (M^e)^2 + \frac{1}{\sqrt{R_s^e\, T^e}}\, g\, s_{lope}^e \right] \begin{pmatrix} -1 \\ 1 \end{pmatrix} \\ &= \left[\frac{1}{8}\, \theta^e\, \sqrt{R_s^e\, T^e}\, |R_+^e - R_-^e|\, \left(R_+^e - R_-^e\right) + \frac{1}{\sqrt{R_s^e\, T^e}}\, g\, s_{lope}^e \right] \begin{pmatrix} -1 \\ 1 \end{pmatrix}. \end{aligned}$$

4. The Node Conditions for the Network Flow. In this section we introduce the coupling conditions that model the flow through the nodes of the network. The node conditions that determine the flow dynamics are given in [2] for the case that all pipes have the same diameter D^e. Let $G = (V, E)$ be a finite connected directed graph. Each edge $e \in E$ of the graph corresponds to an interval $[0, L^e]$. At the vertices $v \in V$, the flow is governed by the node conditions that require the conservation of mass and the continuity of the pressure. Let $E_0(v)$ denote the set of edges in the graph that are incident to $v \in V$ and $x^e(v) \in \{0, L^e\}$ denote the end of the interval $[0, L^e]$ that corresponds to the edge e that is adjacent to v. Define

$$\sigma(v,\, e) = \begin{cases} -1 & \text{if} \quad x^e(v) = 0 \text{ and } \ e \in E_0(v), \\ \ \ 1 & \text{if} \quad x^e(v) = L^e \text{ and } \ e \in E_0(v), \\ \ \ 0 & \text{if} \quad e \notin E_0(v). \end{cases}$$

The incidence matrix $A \in \mathbb{R}^{|V| \times |E|}$ corresponding to G is defined by

$$A = (\, \sigma(v,\, e)\,)_{v \in V,\, e \in E}.$$

The continuity of the pressure at v means that for all $e,\ f \in E_0(v)$ we require the equation

$$p^e(x^e(v)) = p^f(x^f(v)). \tag{21}$$

Moreover, we assume that the Kirchhoff condition

$$\sum_{e \in E_0(v)} \sigma(v,\, e)\, (D^e)^2\, q^e(x^e(v)) = \frac{4}{\pi} b^v \tag{22}$$

holds, where b^v is the mass source flow at v. Define the mass flows $m^e = \frac{\pi}{4}(D^e)^2 q^e$. Then $q^e = \frac{4}{\pi (D^e)^2}\, m^e$. Define the set

$$B = \Big\{ b \in \mathbb{R}^{|V|} : \sum_{v \in V} b_v = 0 \Big\}.$$

For $b \in B$, the Kirchhoff conditions can be written as

$$A\, m = b, \quad m \in \mathbb{R}^{|E|}.$$

4.1. The node conditions in terms of Riemann invariants. In order to state our initial boundary value problem completely in terms of Riemann invariants, we also state the node conditions in terms of Riemann invariants. Since (21) implies that $\ln(p^e(x^e(v))) = \ln(p^f(x^f(v)))$, due to (11), in terms of the Riemann invariants, (21) is equivalent to the linear equation

$$(R_+^e + R_-^e)(x^e(v)) = (R_+^f + R_-^f)(x^f(v)) \tag{23}$$

for all $e,\, f \in E_0(v)$. Note that due to (4), if for all $e,\, f \in E_0(v)$ we have $T^e = T^f$ and $\alpha^e = \alpha^f$, (21) implies that for all $e,\, f \in E_0(v)$ we have $c^e = c^f$ at v, that is the sound speed is also continuous at the node. Due to (5) and (10) we have

$$q^e = \frac{1}{\sqrt{R_s^e\, T^e}}\, p^e\, M^e = \frac{1}{\sqrt{R_s^e\, T^e}}\, \frac{p^e}{c^e} \sqrt{R_s^e\, T^e} \frac{R_+^e - R_-^e}{2} = \frac{p^e}{c^e}\, \frac{R_+^e - R_-^e}{2}.$$

For all $f \in E_0(v)$ can write (22) in the form

$$\sum_{e \in E_0(v)} \sigma(v,\, e)\, (D^e)^2\, \frac{p^f}{c^f}\, \frac{R_+^e - R_-^e}{2} (x^e(v)) = \frac{4}{\pi} b^v. \tag{24}$$

If $b^v = 0$, after division by p^f/c^f which is uniquely determined at the node v this yields the linear equation

$$\sum_{e \in E_0(v)} \sigma(v,\, e)\, (D^e)^2\, \frac{R_+^e - R_-^e}{2} (x^e(v)) = 0. \tag{25}$$

For a boundary node v and the uniquely determined $e \in E_0(v)$ we state the boundary conditions in terms of Riemann invariants in the form

$$R_+^e(x^e(v)) = u_+^v \text{ if } x^e(v) = 0, \tag{26}$$

$$R_-^e(x^e(v)) = u_-^v \text{ if } x^e(v) = L^e \tag{27}$$

with boundary data u_+^v, u_-^v respectively.

Now we assume that the case $b^v \neq 0$ can only occur at the boundary nodes where $|E_0(v)| = 1$. Then for all $v \in V$ the node conditions (23), (25), (26), (27) can be written in the form of the affine linear equation

$$R_{out}^v = \Omega^v R_{in}^v + u^v \tag{28}$$

where $\Omega^v \in \mathbb{R}^{E_0(v) \times E_0(v)}$ is an appropriately chosen matrix and $R_{in}^v \in \mathbb{R}^{E_0(v)}$ is constructed in the following manner: If $e \in E_0(v)$ and $\sigma(v,\, e) = 1$, $R_+^e(x^e(v))$ is a component of R_{in}^v and if $e \in E_0(v)$ and $\sigma(v,\, e) = -1$, R_{in}^v contains $R_-^e(x^e(v))$ as a component.

Moreover, if $e \in E_0(v)$ and $\sigma(v,\, e) = 1$, $R_-^e(x^e(v))$ is a component of R_{out}^v and if $e \in E_0(v)$ and $\sigma(v,\, e) = -1$, R_{out}^v contains $R_+^e(x^e(v))$ as a component. Thus the values of $R_\pm^e(x^e(v))$ $(e \in E_0(v))$ that are not contained in R_{in}^v are collected in R_{out}^v. For an interior node, $u^v \in \mathbb{R}^{E_0(v)}$ is zero and for a boundary node $\Omega^v \in \mathbb{R}$ is zero and u contains the corresponding boundary data.

In fact, for the interior nodes where $|E_0(v)| \geqslant 2$, (23) and (25) imply that

$$R^v_{out} = -R^v_{in} + \mu^v(R^v_{in})\not\Vdash \tag{29}$$

where $\not\Vdash \in \mathbb{R}^{|E_0(v)|}$ is the vector where each component is equal to 1 and the linear functional $\mu^v(R^v_{in})$ is given by the equation

$$\mu^v(R^v_{in}) = 2\,\frac{\sum\limits_{e\in E_0(v)} (D^e)^2\,(R^v_{in})_e}{\sum\limits_{f\in E_0(v)} (D^f)^2}. \tag{30}$$

Thus we have

$$\Omega^v = -Id + \frac{2}{\sum\limits_{g\in E_0(v)} (D^g)^2}\ \mathrm{diag}\ \left((D^e)^2\right)_{e\in E_0(v)} \tag{31}$$

where Id denotes the $|E_0(v)| \times |E_0(v)|$ identity matrix. Since Ω^v is a diagonal matrix and the diagonal elements are in the interval $[-1,\,1]$, we have

$$\|\Omega^v\|_\infty \leqslant 1 \tag{32}$$

where $\|\cdot\|_\infty$ denotes the row-sum norm.

Let Ω^e_v denote the row of the matrix Ω^v that corresponds to the edge e.

4.2. An initial boundary value problem for the Euler equations. In this section we prove an existence theorem for the isothermal Euler equations on a network with the node conditions that we have discussed in Section 4.1 that guarantee the conservation of mass an the continuity of the pressure. For this purpose, the system is written in terms of Riemann invariants in order to obtain a quasilinear system in diagonal form. For our analysis it is useful that in the interior nodes, the node conditions are given by a linear map.

For $\tilde{L}_M > 0$, $u_{\max} > 0$, $\kappa > 0$ and $e \in E$ define the set

$$M^e_2(\tilde{L}_M) = \Big\{(R_+,\,R_-) \in M_1(u_{\max}) :\ \left|R^e_+(t,x) - R^e_-(t,x)\right| \leqslant 2\,\kappa \text{ and } \\ R_+ \text{ and } R_- \text{ are continuous on } [0,\,T]\times[0,\,L^e] \\ \text{and Lipschitz continuous with respect to } x \text{ on } [0,T]\times[0,L^e] \\ \text{with the Lipschitz constant } \tilde{L}_M\Big\}. \tag{33}$$

For the statement of Theorem 2, we introduce the following notation for the source term:

$$\sigma^e(R^e_+,\,R^e_-) = \frac{1}{8}\,\theta^e\,\sqrt{R^e_s\,T^e}\,|R^e_+ - R^e_-|\,(R^e_+ - R^e_-) + \frac{1}{\sqrt{R^e_s\,T^e}}\,g\,s^e_{lope}. \tag{34}$$

Moreover, we define the number

$$\sigma_{\max}(\kappa) = \max_{e\in E}\,\frac{1}{2}\,\theta^e\,\sqrt{R^e_s\,T^e}\,\kappa^2. \tag{35}$$

Let L_σ denote a common Lipschitz constant for the functions σ^e on the sets $M^e_2(\tilde{L}_M)$ in the sense that for all $e \in E$ for all $(R^e_+, R^e_-), (S^e_+, S^e_-) \in M^e_2(\tilde{L}_M)$ we have the Lipschitz inequality

$$|\sigma^e(R^e_+,\,R^e_-) - \sigma^e(S^e_+,\,S^e_-)| \leqslant L_\sigma\ \left(|R^e_+ - S^e_+| + |S^e_+ - S^e_-|\right). \tag{36}$$

In fact we can choose

$$L_\sigma = \frac{1}{2}\,\theta^e\,\sqrt{R_s^e\,T^e}\,\kappa. \tag{37}$$

THEOREM 2. *Assume that for all $e \in E$, $s^e_{lope} = 0$ that is we consider a network of horizontal pipes. Assume that T^e and α^e are independent of $e \in E$. Define the diagonal system matrix*

$$D^e = \begin{pmatrix} \lambda_+^e & 0 \\ 0 & \lambda_-^e \end{pmatrix}$$

$$= \sqrt{R_s^e\,T^e}\left[\left(1+\alpha^e \exp\left(\tfrac{R_+^e+R_-^e}{2}\right)\right)\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} + \tfrac{R_+^e-R_-^e}{2}\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}\right].$$

Consider the system

$$(\mathbf{S})\begin{cases} R_+^e(0,x) = y_+^e(x),\ x\in(0,L^e),\ e\in E,\\ R_-^e(0,x) = y_-^e(x),\ x\in(0,L^e),\ e\in E,\\ R_+^e(t,0) = u_+^v(t),\ t\in(0,T) \text{ if } \ 0 = x^e(v) \text{ and } \ |E_0(v)| = 1,\\ R_-^e(t,L^e) = u_-^v(t),\ t\in(0,T) \text{ if } \ L^e = x^e(v) \text{ and } \ |E_0(v)| = 1,\\ R_+^e(t,0) = \Omega_v^e\,R_{in}^v(t),\ t\in(0,T) \text{ if } \ 0 = x^e(v) \text{ and } \ |E_0(v)| \geqslant 2,\\ R_-^e(t,L^e) = \Omega_v^e\,R_{in}^v(t),\ t\in(0,T) \text{ if } \ L^e = x^e(v) \text{ and } \ |E_0(v)| \geqslant 2,\\ \begin{pmatrix} R_+^e \\ R_-^e \end{pmatrix}_t + D^e\begin{pmatrix} R_+^e \\ R_-^e \end{pmatrix}_x = \left[\tfrac{\theta^e}{8}\ \sqrt{R_s^e\,T^e}\,|R_+^e - R_-^e|\,(R_+^e - R_-^e)\right]\begin{pmatrix} -1 \\ 1 \end{pmatrix}\\ \text{on } \ [0,T]\times[0,L^e],\ e\in E. \end{cases}$$

Define the number

$$u_{\max} = \min_{e\in E}\left\{\frac{1}{2},\ -\ln(2\,|\alpha^e|)\right\}. \tag{38}$$

Let

$$T \in \left(0,\ \min_{e\in E}\frac{L^e}{\Lambda^e(u_{\max})}\right) \tag{39}$$

be given. Assume that the C^0-compatibility conditions are satisfied between y_+^e and u_+^e (that is $y_+^e(0) = u_+^e(0)$) and between y_-^e and u_-^e (that is $y_-^e(L^e) = u_-^e(0)$) and at the interior nodes (that is $R_{out}^v(0) = \Omega^v\,R_{in}^v(0)$ for all $v \in V$ with $|E_0(v)| \geqslant 2$). Assume that y_+^e and u_+^e and y_-^e and u_-^e are Lipschitz continuous. Let

$$\tilde{L}_R$$

denote a common Lipschitz constant for u_+^e, y_+^e on $[0,T]$, $[0,L^e]$ respectively and u_-^e, y_-^e on $[0,T]$, $[0,L^e]$ respectively such that for all $e \in E$ we also have

$$|u_+^e(t) - y_+^e(x)| \leqslant \tilde{L}_R\,(|t| + |x|), \tag{40}$$

$$|u_-^e(t) - y_-^e(x)| \leqslant \tilde{L}_R\,(|t| + |L^e - x|) \tag{41}$$

for all $t \in [0,T]$ and $x \in [0,\,L^e]$. Define the numbers

$$B_\pm^e = \sup_{(t,x)\in[0,T]\times[0,L^e]}\{|u_\pm^e(t)|,\ |y_\pm^e(x)|\}, \qquad B_{\max}(T) = \max_{e\in E}\{B_+^e,\ B_-^e\}. \tag{42}$$

We assume that there exists a number $\kappa > 0$ such that for all $e \in E$ we have

$$B_{\max}(T) + \frac{1}{2}\, T\, \theta^e \sqrt{R_s^e\, T^e}\, \kappa^2 \leqslant \min\{u_{\max},\, \kappa\}. \tag{43}$$

If there is a number $\tilde{L}_M \geqslant \tilde{L}_R$ such that

$$\tilde{L}_R \left[\max_{e\in E} \frac{\mathrm{e}^{2\,\Lambda^e(u_{\max})\,\tilde{L}_M\,T}}{\underline{\Lambda}^e(u_{\max})} \left(1 + \max_{e\in E} \Lambda^e(u_{\max})\right) + 2\, \max_{e\in E} \mathrm{e}^{2\,\Lambda^e(u_{\max})\,\tilde{L}_M\,T} \right] \tag{44}$$

$$+\, 2 \left[2\,T\,L_\sigma\,\tilde{L}_M\, \max_{e\in E} \mathrm{e}^{2\,\Lambda^e(u_{\max})\,\tilde{L}_M\,T} + \sigma_{\max}(\kappa)\, \max_{e\in E} \frac{\mathrm{e}^{2\,\Lambda^e(u_{\max})\,\tilde{L}_M\,T}}{\underline{\Lambda}^e(u_{\max})} \right] \leqslant \tilde{L}_M$$

with L_σ as defined in (36) and $\sigma_{\max}(\kappa)$ as defined in (35) and

$$\max_{e\in E} \Big[2\,T\,L_\sigma\, \left(2 + \tilde{L}_M\,T\,\Lambda^e(u_{\max})\, \exp(2\Lambda^e(u_{\max})\,\tilde{L}_M\,T)\right) \tag{45}$$

$$+\, T\,\Lambda^e(u_{\max})\,\mathrm{e}^{2\Lambda^e(u_{\max})\,\tilde{L}_M\,T} \left(\tilde{L}_R + \frac{\tilde{L}_R}{\underline{\Lambda}^e(u_{\max})} + 2\,T\,L_\sigma\,\tilde{L}_M \right) \Big] < 1,$$

then $(\mathbf{S})$ *has a unique solution on $[0, T]$ that solves the initial boundary value problem in the sense of characteristics, that is it satisfies the corresponding integral equations along the characteristic curves. For all $e \in E$, we have $(R_+^e,\, R_-^e) \in M_2^e(\tilde{L}_M)$ with the set $M_2^e(\tilde{L}_M)$ defined in (33). For all $e \in E$ and $(t,\, x) \in [0, T] \times [0, L^e]$, the solution satisfies the a-priori inequality*

$$\left|R_+^e(t,\, x)\right| + \left|R_-^e(t,\, x)\right| \leqslant 2\, \exp\left(\frac{1}{2}\,\theta^e\, \sqrt{R_s^e\,T^e}\,\kappa\,T\right)\, B_{\max}(T). \tag{46}$$

The solution is Lipschitz continuous with respect to x, that is for all $e \in E$, the functions $R_\pm^e$ are Lipschitz continuous on $[0,\, T] \times [0,\, L^e]$ with the Lipschitz constant L_T that satisfies the a-priori inequality

$$L_T \leqslant \max_{e\in E} \frac{\mathrm{e}^{2\,\Lambda^e(u_{\max})\,\tilde{L}_M\,T}}{\underline{\Lambda}^e(u_{\max})} \left[\tilde{L}_R \left(1 + \max_{e\in E} \Lambda^e(u_{\max})\right) + 2\,\sigma_{\max}(\kappa) \right] \tag{47}$$

$$+ \max_{e\in E} \mathrm{e}^{2\,\Lambda^e(u_{\max})\,\tilde{L}_M\,T} \left(2\,\tilde{L}_R + 4\,T\,L_\sigma\,\tilde{L}_M\right).$$

Note that L_T can be made arbitrarily small by choosing $\tilde{L}_R$ and κ (and hence L_σ due to (37) and $\sigma_{\max}(\kappa)$ due to (35)) sufficiently small.

REMARK 1. *Theorem 2 implies that for a given time interval $[0,\, T]$ where T satisfies (39) the solution of the initial boundary value problem exists, if $\tilde{L}_R$ and the values of θ^e $(e \in E)$ (and hence L_σ) are sufficiently small.*

Theorem 2 also implies that for given values of θ^e $(e \in E)$ the solution of the initial boundary value problem exists on $[0,\, T]$, if $\tilde{L}_R$, κ and T are sufficiently small.

In Step 2 of the proof of Theorem 2, Condition (44) implies that the solution is Lipschitz continuous with respect to x with a Lipschitz constant that is less than or equal to $\tilde{L}_M$.

Remark 2. *Note that the inequality (44) holds if $T > 0$ is chosen sufficiently small and*

$$\tilde{L}_M \geqslant \tilde{L}_R \left[3 + \max_{e \in E} \frac{1}{\underline{\Lambda}^e(u_{\max})} \left(1 + \max_{e \in E} \Lambda^e(u_{\max}) + 2\, \sigma_{\max}(\kappa) \right) \right].$$

In addition, T can be chosen so small that inequality (45) also holds.

The proof of Theorem 2 is based upon the classical idea of a fixed point iteration along the characteristic curves. For this purpose, let us define the characteristic curves for given $\mathcal{R}^e = (R_+^e,\, R_-^e) \in M_e^2(\tilde{L}_M)$. For $s \in [0,\, T]$, $x \in [0,\, L^e]$, $t \in [0,\, T]$ we define $\xi_\pm^{\mathcal{R}^e}(s, x, t)$ as the solution of the initial value problem

$$\xi_\pm^{\mathcal{R}^e}(t, x, t) = x,\ \partial_s \xi_\pm^{\mathcal{R}^e}(s, x, t) = \lambda_\pm^e(\mathcal{R}^e)$$

thus this characteristic curve runs through the point (t, x) in $[0,\, T] \times [0, L^e]$.

The characteristic curves $\xi_+^{\mathcal{R}^e}(s, x, t)$ can arrive at the boundary $x = L^e$. Then they are reflected and continue as the $\xi_-^{\mathcal{R}^e}$-characteristic that starts at the reflection point. On the other hand the characteristic curves $\xi_-^{\mathcal{R}^e}(s, x, t)$ can arrive at the boundary $x = 0$ where they are reflected and continue as the $\xi_+^{\mathcal{R}^e}$-characteristic that starts at the reflection point. In this way a family of curves is generated that are well-defined for all $t \in [0, T]$. If T is sufficiently large, the curves zigzag between $x = 0$ and $x = L^e$.

Due to the structure of the diagonal matrix D^e we have for all $e \in E$, $x \in (0, L^e)$ and t in a neighborhood of s the integral equation

$$\xi_\pm^{\mathcal{R}^e}(s, x, t) = x \pm (R_s^e T^e)^{\frac{1}{2}} \left[(s - t) + \int_t^s \alpha^e \exp\left(\frac{R_+^e + R_-^e}{2} \right) (\tau,\, \xi_\pm^{\mathcal{R}^e}(\tau,\, x,\, t))\, d\tau \right]$$
$$+ (R_s^e T^e)^{\frac{1}{2}} \int_t^s \left(\frac{R_+^e - R_-^e}{2} \right) (\tau,\, \xi_\pm^{\mathcal{R}^e}(\tau,\, x,\, t))\, d\tau \in [0, L].$$

In the following Lemma we give a sufficient condition for the existence of the family of characteristic curves for given $\mathcal{R}^e = (R_+^e, R_-^e)$. This is essentially a consequence of the classical Picard-Lindelöf Theorem. However, we have to adapt it to the situation of our initial boundary value problem. In particular, we have to take care of the interplay between the boundary conditions and the initial conditions.

We also provide Lipschitz constants for the characteristic curves as functions of their input data, in particular as functions of the generating pair of Riemann invariants.

Lemma 3. *Assume that $\mathcal{R}^e = (R_+^e,\, R_-^e) \in C([0, T] \times [0, L^e])^2$ is Lipschitz continuous with respect to x with the Lipschitz–constant $L_{\mathcal{R}^e}$. Define*

$$u_{\max} = \min \left\{ \frac{1}{2},\ -\ln(2\, |\alpha^e|) \right\}. \tag{48}$$

Let $T \in (0,\, \frac{L^e}{\Lambda^e(u_{\max})})$ with $\Lambda^e(u_{\max})$ as defined in (15). Assume that (R_+^e, R_-^e) is in the set $M_1(u_{\max})$ as defined in (14). Then the family of characteristic curves $\xi_\pm^{\mathcal{R}^e}(s, x, t)$ exists for all

$$(s, x, t) \in [0, T] \times [0, L^e] \times [0, T]$$

with at most one reflection at the boundary points $x = 0$ or $x = L^e$. The functions $\xi_\pm^{\mathcal{R}^e}(s,x,t)$ are continuously differentiable with respect to s and the derivative is Lipschitz continuous with the Lipschitz constant $\Lambda^e(u_{\max})\, L_{\mathcal{R}^e}$.

For $\mathcal{S}^e = (S_+^e,\, S_-^e) \in C([0,T]\times[0,L])$ that is Lipschitz continuous with respect to x with the Lipschitz–constant $L_{\mathcal{R}^e}$ we have the inequality

$$|\xi_\pm^{\mathcal{R}^e}(s,x,t) - \xi_\pm^{\mathcal{S}^e}(s,x,t)| \tag{49}$$

$$\leqslant \frac{\exp(2\, L_{\mathcal{R}^e}\, s\, \Lambda^e(u_{\max})) - 1}{2\, L_{\mathcal{R}^e}} \, \|\mathcal{R}^e - \mathcal{S}^e\|_{C([0,T]\times[0,L^e])^2} \tag{50}$$

$$\leqslant T\, \Lambda^e(u_{\max})\, \exp\left(2\, L_{\mathcal{R}^e}\, T\, \Lambda^e(u_{\max})\right)\, \|\mathcal{R}^e - \mathcal{S}^e\|_{C([0,T]\times[0,L^e])^2} \tag{51}$$

where the norm is defined as

$$\|\mathcal{R}^e - \mathcal{S}^e\|_{C([0,T]\times[0,L^e])^2} = \max_{(t,x)\in[0,T]\times[0,L^e]} |R_+^e(t,x) - S_+^e(t,x)| + |R_-^e(t,x) - S_-^e(t,x)|.$$

If for all $s \in [0,T]$ we have $\xi_+^{\mathcal{R}^e}(s,\, x,\, t) > 0$, we define $t_+^{\mathcal{R}^e}(x,t) = 0$. Else let $t_+^{\mathcal{R}^e}(x,t) \in [0,T]$ be defined as the uniquely determined time with

$$\xi_+^{\mathcal{R}^e}(t_+^{\mathcal{R}^e}(x,\, t),\, x,\, t) = 0.$$

Analogously we define $t_-^{\mathcal{R}^e}(x,\, t) = 0$ if for all $s \in [0,T]$ we have $\xi_-^{\mathcal{R}^e}(s,\, x,\, t) < L^e$. Otherwise let $t_-^{\mathcal{R}^e}(x,\, t)$ denote the unique solution of the equation

$$\xi_-^{\mathcal{R}^e}(t_-^{\mathcal{R}^e}(x,\, t),\, x,\, t) = L^e.$$

Then we have

$$|t_\pm^{\mathcal{R}^e}(x,t) - t_\pm^{\mathcal{S}^e}(x,t)| \leqslant \frac{T\, \Lambda^e(u_{\max})\, \exp(2\, L_{\mathcal{R}^e}\, T\, \Lambda^e(u_{\max}))}{\underline{\Lambda}^e(u_{\max})} \, \|\mathcal{R}^e - \mathcal{S}^e\|_{C([0,T]\times[0,L^e])^2} \tag{52}$$

with $\underline{\Lambda}^e(u_{\max})$ as defined in (17). For $x_1,\, x_2 \in [0,L]$, $s,\, t \in [0,T]$ we have

$$|\xi_\pm^{\mathcal{R}^e}(s,\, x_1,t) - \xi_\pm^{\mathcal{R}^e}(s,\, x_2,t)| \leqslant \exp(2\, L_{\mathcal{R}^e} s\, \Lambda^e(u_{\max}))\, |x_1 - x_2| \tag{53}$$

and

$$|t_\pm^{\mathcal{R}^e}(x_1,t) - t_\pm^{\mathcal{R}^e}(x_2,t)| \leqslant \frac{1}{\underline{\Lambda}^e(u_{\max})}\, \exp(2\, L_{\mathcal{R}^e}\, T\, \Lambda^e(u_{\max}))\, |x_1 - x_2|. \tag{54}$$

Proof of Lemma 3: Due to (48), (18) holds.

We define a fixed point iteration for $\xi_\pm^{\mathcal{R}^e}(s,x,t)$. Here we want to avoid the reflections at $x = 0$ and $x = L^e$ by extending R_+^e and R_-^e continuously on the whole x-axis. For this purpose we define $R_\pm^e(t,x) = R_\pm^e(t,0)$ if $x < 0$ and $R_\pm^e(t,x) = R_\pm^e(t,L^e)$ if $x > L^e$. Then the extension of $R_\pm^e$ is continous on $[0,T]\times\mathbb{R}$ and Lipschitz continuous with respect to x with the Lipschitz constant $L_{\mathcal{R}^e}$. We consider the integral equation

$$\xi_\pm^{\mathcal{R}^e}(s,x,t) = x + \int_t^s \lambda_\pm^e(R_+^e, R_-^e)(\tau,\, \xi_\pm^{\mathcal{R}^e}(\tau,\, x,\, t))\, d\tau \tag{55}$$

for $(s,x,t) \in [0,T]\times\mathbb{R}\times[0,T]$ where $\lambda_\pm^e(R_+^e, R_-^e)$ is defined as in (12), (13) respectively. To show the existence of a unique solution, we look at the Picard-Lindelöf iteration with the starting point

$$\xi_\pm^{(1)}(s,x,t) = x$$

and the successive iterates (for $k \in \{1, 2, 3, ...\}$)

$$\xi_{\pm}^{(k+1)}(s,x,t) = x + \int_t^s \left(\lambda_{\pm}^e(R_+^e, R_-^e)\right)(\tau,\, \xi_{\pm}^{(k)}(\tau,\, x,\, t))\, d\tau.$$

For $k \in \{2, 3, 4...\}$ this implies due to the Lipschitz inequality (16) in Lemma 1

$$\begin{aligned}
&\left|\xi_{\pm}^{(k+1)}(s,x,t) - \xi_{\pm}^{(k)}(s,x,t)\right| \\
&= \left|\int_t^s \lambda_{\pm}^e(R_+^e, R_-^e)(\tau,\, \xi_{\pm}^{(k)}(\tau,\, x,\, t)) - \lambda_{\pm}^e(R_+^e, R_-^e)(\tau,\, \xi_{\pm}^{(k-1)}(\tau,\, x,t))\, d\tau\right| \\
&\leqslant 2\,\Lambda^e(u_{\max})\, L_{\mathcal{R}^e} \left|\int_t^s |\xi_{\pm}^{(k)}(\tau,\, x,\, t) - \xi_{\pm}^{(k-1)}(\tau,\, x,\, t)|\, d\tau\right|.
\end{aligned}$$

We have

$$\begin{aligned}
\left|\xi_{\pm}^{(2)}(s,x,t) - \xi_{\pm}^{(1)}(s,x,t)\right| &= \left|\int_t^s \lambda_{\pm}^e(R_+^e, R_-^e)(\tau,\, x)\, d\tau\right| \\
&\leqslant \Lambda^e(u_{\max})\, |t-s|.
\end{aligned}$$

By induction this implies the inequality

$$\begin{aligned}
\left|\xi_{\pm}^{(k+1)}(s,x,t) - \xi_{\pm}^{(k)}(s,x,t)\right| &\leqslant \frac{1}{k!}\, \Lambda^e(u_{\max}) \quad \left(2\,\Lambda^e(u_{\max})\, L_{\mathcal{R}^e}\right)^{k-1} \; |t-s|^k \\
&= \frac{1}{k!}\, \frac{\Lambda^e(u_{\max})}{2\,\Lambda^e(u_{\max})\, L_{\mathcal{R}^e}} \left(2\Lambda^e(u_{\max})\, L_{\mathcal{R}^e}\, |t-s|\right)^k
\end{aligned}$$

Since

$$\exp\left(2\Lambda^e(u_{\max})\, L_{\mathcal{R}^e}\, |t-s|\right) = \sum_{k=0}^{\infty} \frac{1}{k!} \; \left(2\Lambda^e(u_{\max})\, L_{\mathcal{R}^e}\, |t-s|\right)^k < \infty$$

this implies that $(\xi_{\pm}^{(k)}(s,x,t))_k$ is a Cauchy sequence in

$$C([0,T] \times \mathbb{R} \times [0,T])$$

and hence convergent. To be precise, we state that for $m > n \geqslant 1$ we have

$$\begin{aligned}
\left|\xi_{\pm}^{(m)}(s,x,t) - \xi_{\pm}^{(n)}(s,x,t)\right| &\leqslant \sum_{k=n}^{m-1} \left|\xi_{\pm}^{(k+1)}(s,x,t) - \xi_{\pm}^{(k)}(s,x,t)\right| \\
&\leqslant \sum_{k=n}^{m-1} \frac{1}{k!}\, \frac{\Lambda^e(u_{\max})}{2\,\Lambda^e(u_{\max})\, L_{\mathcal{R}^e}} \left(2\Lambda^e(u_{\max})\, L_{\mathcal{R}^e}\, |t-s|\right)^k \\
&\to 0 \text{ for } \; m, n \to \infty.
\end{aligned}$$

Hence there exists a limit function $\xi_{\pm}^{\mathcal{R}^e}(s,x,t) \in C([0,T] \times \mathbb{R} \times [0,T])$. This limit function satisfies the integral equation (55). Moreover, as in the proof of the classical Picard-Lindelöf Theorem, we get the uniqueness of the solution of (55).

To get the solution with the reflections, we only have to switch at $x = 0$ from the $+$ to the $-$ characteristic curve and at $x = L^e$ vice versa.

Now we come to (49). The integral equation (55) implies

$$\begin{aligned}
&\left|\xi_\pm^{\mathcal{R}^e}(s,x,t) - \xi_\pm^{\mathcal{S}^e}(s,x,t)\right| \\
&= \left|\int_t^s \lambda_\pm^e(R_+^e, R_-^e)(\tau,\, \xi_\pm^{\mathcal{R}^e}(\tau,\,x,\,t)) - \lambda_\pm^e(S_+^e, S_-^e)(\tau,\, \xi_\pm^{\mathcal{S}^e}(\tau,\,x,\,t))\,d\tau\right| \\
&\leqslant \left|\int_t^s \lambda_\pm^e(R_+^e, R_-^e)(\tau,\, \xi_\pm^{\mathcal{R}^e}(\tau,\,x,\,t)) - \lambda_\pm^e(R_+^e, R_-^e)(\tau,\, \xi_\pm^{\mathcal{S}^e}(\tau,\,x,\,t))\,d\tau\right| \\
&+ \left|\int_t^s \lambda_\pm^e(R_+^e, R_-^e)(\tau,\, \xi_\pm^{\mathcal{S}^e}(\tau,\,x,\,t)) - \lambda_\pm^e(S_+^e, S_-^e)(\tau,\, \xi_\pm^{\mathcal{S}^e}(\tau,\,x,\,t))\,d\tau\right| \\
&\leqslant 2\Lambda^e(u_{\max})\,L_{\mathcal{R}^e}\ \left|\int_t^s \left|\xi_\pm^{\mathcal{R}^e}(\tau,\,x,\,t) - \xi_\pm^{\mathcal{S}^e}(\tau,\,x,\,t)\right|\,d\tau\right| \\
&+\ |t-s|\ \Lambda^e(u_{\max})\,\|\mathcal{R}^e - \mathcal{S}^e\|_{C([0,T]\times[0,L^e])^2}.
\end{aligned}$$

Define

$$U(s) = \left|\xi_\pm^{\mathcal{R}^e}(s,x,t) - \xi_\pm^{\mathcal{S}^e}(s,x,t)\right|.$$

Then we have the integral inequality

$$U(s) \leqslant \left|\int_t^s 2\Lambda^e(u_{\max})\,L_{\mathcal{R}^e}\ U(\tau)\ + \Lambda^e(u_{\max})\,\|\mathcal{R}^e - \mathcal{S}^e\|_{C([0,T]\times[0,L^e])^2}\,d\tau\right|.$$

Now we can apply Gronwall's Lemma (see for example Lemma 1 in [9]). We get

$$U(s) \leqslant \|\mathcal{R}^e - \mathcal{S}^e\|_{C([0,T]\times[0,L^e])^2}\ \frac{\exp(2\,\Lambda^e(u_{\max})\,L_{\mathcal{R}^e}\,s) - 1}{2\,L_{\mathcal{R}^e}}.$$

Thus we have shown (49).

Now we come to (52). Without loss of generality we assume that

$$t_+^{\mathcal{R}^e}(x,t) > t_+^{\mathcal{S}^e}(x,t) \geqslant 0.$$

We have

$$\begin{aligned}
\xi_+^{\mathcal{R}^e}(t_+^{\mathcal{R}^e}(x,t),\,x,\,t) - \xi_+^{\mathcal{R}^e}(t_+^{\mathcal{S}^e}(x,t),\,x,\,t) &= \int_{t_+^{\mathcal{S}^e}(x,t)}^{t_+^{\mathcal{R}^e}(x,t)} \partial_s \xi_+^{\mathcal{R}^e}(s,\,x,\,t)\,ds \\
&= \int_{t_+^{\mathcal{S}^e}(x,t)}^{t_+^{\mathcal{R}^e}(x,t)} \lambda_+^e(R_+^e, R_-^e)(\tau,\, \xi_+^{\mathcal{R}^e}(\tau,\,x,\,t))\,ds \\
&\geqslant \underline{\Lambda}^e(u_{\max})\ \left[t_+^{\mathcal{R}^e}(x,t) - t_+^{\mathcal{S}^e}(x,t)\right].
\end{aligned}$$

On the other hand we have

$$\begin{aligned}
&\xi_+^{\mathcal{R}^e}(t_+^{\mathcal{R}^e}(x,t),\,x,\,t) - \xi_+^{\mathcal{R}^e}(t_+^{\mathcal{S}^e}(x,t),\,x,\,t) \\
&= 0 - \xi_+^{\mathcal{R}^e}(t_+^{\mathcal{S}^e}(x,t),\,x,\,t) \\
&= \xi_+^{\mathcal{S}^e}(t_+^{\mathcal{S}^e}(x,t),\,x,\,t) - \xi_+^{\mathcal{R}^e}(t_+^{\mathcal{S}^e}(x,t),\,x,\,t) \\
&\leqslant T\,\Lambda^e(u_{\max})\ \exp(2\Lambda^e(u_{\max})\,L_{\mathcal{R}^e}\,T)\,\|\mathcal{R}^e - \mathcal{S}^e\|_{C([0,T]\times[0,L^e])^2},
\end{aligned}$$

where the last inequality follows from (49). Thus we have shown (52) for $x = 0$ (that is for the $+$-case).

Now we look at the case $x = 0$ (the minus-case). Again without loss of generality we assume that

$$t_-^{\mathcal{R}^e}(x,t) > t_-^{\mathcal{S}^e}(x,t) \geqslant 0.$$

We have

$$\begin{aligned}\xi_-^{\mathcal{R}^e}(t_-^{\mathcal{R}^e}(x,t),\,x,\,t) - \xi_-^{\mathcal{R}^e}(t_-^{\mathcal{S}^e}(x,t),\,x,\,t) &= \int_{t_-^{\mathcal{S}^e}(x,t)}^{t_-^{\mathcal{R}^e}(x,t)} \partial_s \xi_-^{\mathcal{R}^e}(s,\,x,\,t)\,ds \\ &= \int_{t_-^{\mathcal{S}^e}(x,t)}^{t_-^{\mathcal{R}^e}(x,t)} \lambda_-^e(R_+^e, R_-^e)(\tau,\,\xi_-^{\mathcal{R}^e}(\tau,\,x,\,t))\,ds \\ &\leqslant -\underline{\Lambda}^e(u_{\max})\,\left[t_-^{\mathcal{R}^e}(x,t) - t_-^{\mathcal{S}^e}(x,t)\right].\end{aligned}$$

On the other hand we have

$$\begin{aligned}&\xi_-^{\mathcal{R}^e}(t_-^{\mathcal{S}^e}(x,t),\,x,\,t) - \xi_-^{\mathcal{R}^e}(t_-^{\mathcal{R}^e}(x,t),\,x,\,t) \\ &= \xi_-^{\mathcal{R}^e}(t_-^{\mathcal{S}^e}(x,t),\,x,\,t) - L^e \\ &= \xi_-^{\mathcal{R}^e}(t_-^{\mathcal{S}^e}(x,t),\,x,\,t) - \xi_-^{\mathcal{S}^e}(t_-^{\mathcal{S}^e}(x,t),\,x,\,t) \\ &\leqslant T\,\Lambda^e(u_{\max})\,\exp(2\Lambda^e(u_{\max})\,L_{\mathcal{R}^e}\,T)\,\|\mathcal{R}^e - \mathcal{S}^e\|_{C([0,T]\times[0,L^e])^2},\end{aligned}$$

where again the last inequality follows from (49). Thus we have shown (52) for $x = L^e$ (that is for the minus–case).

Inequality (53) follows with Gronwall's Lemma.

Now we show (54) for the plus–case. Let $x_1,\,x_2 \in [0, L^e]$ be given. Without loss of generality we assume that $x_1 < x_2$. Then we get

$$t_+^{\mathcal{R}^e}(x_1,t) \geqslant t_+^{\mathcal{R}^e}(x_2,t).$$

Case 1: If $t_+^{\mathcal{R}^e}(x_1,t) = 0$, we have $t_+^{\mathcal{R}^e}(x_2,t) = 0$ and thus $t_+^{\mathcal{R}^e}(x_1,t) - t_+^{\mathcal{R}^e}(x_2,t) = 0$.

Case 2.: If $t_+^{\mathcal{R}^e}(x_1,t) > 0$, we get

$$\xi_+^{\mathcal{R}^e}(t_+^{\mathcal{R}^e}(x_1,t),\,x_1,\,t) = 0.$$

This yields

$$\begin{aligned}\xi_+^{\mathcal{R}^e}(t_+^{\mathcal{R}^e}(x_1,t),\,x_1,\,t) - \xi_+^{\mathcal{R}^e}(t_+^{\mathcal{R}^e}(x_2,t),\,x_1,\,t) &= \int_{t_+^{\mathcal{R}^e}(x_2,t)}^{t_+^{\mathcal{R}^e}(x_1,t)} \partial_s \xi_+^{\mathcal{R}^e}(s,\,x_1,\,t)\,ds \\ &\geqslant \underline{\Lambda}^e(u_{\max})\,\left[t_+^{\mathcal{R}^e}(x_1,t) - t_+^{\mathcal{R}^e}(x_2,t)\right].\end{aligned}$$

On the other hand we have

$$\begin{aligned}&\xi_+^{\mathcal{R}^e}(t_+^{\mathcal{R}^e}(x_1,t),\,x_1,\,t) - \xi_+^{\mathcal{R}^e}(t_+^{\mathcal{R}^e}(x_2,t),\,x_1,\,t) \\ &= 0 - \xi_+^{\mathcal{R}^e}(t_+^{\mathcal{R}^e}(x_2,t),\,x_1,\,t) \\ &\leqslant \xi_+^{\mathcal{R}^e}(t_+^{\mathcal{R}^e}(x_2,t),\,x_2,\,t) - \xi_+^{\mathcal{R}^e}(t_+^{\mathcal{R}^e}(x_2,t),\,x_1,\,t) \leqslant \exp(2\,\Lambda^e(u_{\max})\,L_{\mathcal{R}^e}\,T)\,|x_1 - x_2|\end{aligned}$$

where the last inequality follows from (53). Hence we have shown (54) for the +-case. The inequality for the minus-case can be shown analogously.

Thus Lemma 3 is proven. □

Now we come to the proof of the existence of a unique solution of the initial boundary value problem (**S**), that is based upon a fixed point iteration.

Proof of Theorem 2:

For given functions $\mathcal{R}^e = (R_+^e,\, R_-^e) \in M_2(\tilde{L}_M)$, Lemma 3 implies the existence of characteristic fields $\xi_\pm^{\mathcal{R}^e}$ and the corresponding functions $t_\pm^{\mathcal{R}^e}$. Due to (39), for all $e \in E$ the characteristic curves cannot go from their starting point to the other end of the space interval $[0,\, L^e]$. For all $e \in E$, on the set $M_2^e(\tilde{L}_M)$ we define the following mapping:

For given $\mathcal{R}^e = (R_+^e,\, R_-^e) \in M_2^e(\tilde{L}_M)$ we define

$$\Phi_+^e(R_+^e,\, R_-^e)(t,x) = R_+^e(t_+^{\mathcal{R}^e}(x,t),\, \xi_+^{\mathcal{R}^e}(t_+^{\mathcal{R}^e}(x,t),\, x,\, t)) - \int_{t_+^{\mathcal{R}^e}(x,t)}^t \sigma^e(R_+^e,\, R_-^e)\, \left(\tau,\, \xi_+^{\mathcal{R}^e}(\tau,\, x,\, t)\right)\, d\tau$$

with σ^e as defined in (34). Here we have

$$R_+^e(t_+^{\mathcal{R}^e}(x,t),\, \xi_+^{\mathcal{R}^e}(t_+^{\mathcal{R}^e}(x,t),\, x,\, t))$$

$$= \begin{cases} u_+^e(t_+^{\mathcal{R}^e}(x,t)) & \text{if} \quad t_+^{\mathcal{R}^e}(x,t) > 0,\, 0 = x^e(v) \text{ and } |E_0(v)| = 1, \\ y_+^e(\xi_+^{\mathcal{R}^e}(0,\, x,\, t)) & \text{if} \quad t_+^{\mathcal{R}^e}(x,t) = 0, \\ \Omega_v^e\, R_{in}^v(t) & \text{if} \quad t_+^{\mathcal{R}^e}(x,t) > 0,\, 0 = x^e(v) \text{ and } |E_0(v)| \geqslant 2. \end{cases}$$

The components of $R_{in}^v(t)$ that appear in the last line are in turn obtained by integrating along the characteristic curves $\xi_\pm^{\mathcal{R}^e}$. Due to (39), they can be followed back to the initial state, that is for $f \in E_0(v)$ the components of $R_{in}^v(t)$ have the form

$$R_\pm^f(t,\, x^f(v)) = y_\pm^f(\xi_\pm^{\mathcal{R}^f}(0,\, x,\, t)) \mp \int_0^t \sigma^f(R_+^f,\, R_-^f)\, \left(\tau,\, \xi_\pm^{\mathcal{R}^f}(\tau,\, x,\, t)\right)\, d\tau \tag{56}$$

without further reflections. Analogously we define

$$\Phi_-^e(R_+^e,\, R_-^e)(t,x) = R_-^e(t_-^{\mathcal{R}^e}(x,t),\, \xi_-^{\mathcal{R}^e}(t_-^{\mathcal{R}^e}(x,t),\, x,\, t)) + \int_{t_-^{\mathcal{R}^e}(x,t)}^t \sigma^e(R_+^e,\, R_-^e)\, \left(\tau,\, \xi_-^{\mathcal{R}^e}(\tau,\, x,\, t)\right)\, d\tau.$$

Here we have

$$R_-^e(t_+^{\mathcal{R}^e}(x,t),\, \xi_-^{\mathcal{R}^e}(\mathcal{R^e}_+(x,t),\, x,\, t))$$

$$= \begin{cases} u_-(t_-^{\mathcal{R}^e}(x,t)) & \text{if} \quad t_-^{\mathcal{R}^e}(x,t) > 0,\, L^e = x^e(v) \text{ and } |E_0(v)| = 1, \\ y_-(\xi_-^{\mathcal{R}^e}(0,\, x,\, t)) & \text{if} \quad t_-^{\mathcal{R}^e}(x,t) = 0, \\ \Omega_v^e\, R_{in}^v(t) & \text{if} \quad t_-^{\mathcal{R}^e}(x,t) > 0,\, L^e = x^e(v) \text{ and } |E_0(v)| \geqslant 2, \end{cases}$$

Again the components of $R_{in}^v(t)$ are obtained by integrating along the corresponding characteristic curves $\xi_\pm^{\mathcal{R}^f}$ for $f \in E_0(v)$ going back to the given initial values as in (56).

In this way we get the fixed point iteration where for all $e \in E$ we define

$$\begin{pmatrix} \rho_+^{e,(k+1)}(t,x) \\ \rho_-^{e,(k+1)}(t,x) \end{pmatrix} = \begin{pmatrix} \Phi_+^e(\rho_+^{e,(k)},\, \rho_-^{e,(k)})(t,x) \\ \Phi_-^e(\rho_+^{e,(k)},\, \rho_-^{e,(k)})(t,x) \end{pmatrix} \tag{57}$$

that we start with functions $(\rho_+^{e,(1)},\, \rho_-^{e,(1)}) \in M_2^e(\tilde{L}_M)$.

Our aim is to apply Banach's fixed point theorem. We check in several steps that the assumptions hold. First we show that the fixed point iteration is well–defined.

Step 1 (The fixed point iteration is well–defined) In order to show that the fixed point iteration is well–defined, we show that the iterates remain in $M_1(u_{\max})$. If $\rho^{e,(k)} = (\rho_+^{e,(k)},\, \rho_-^{e,(k)}) \in M_1(u_{\max})$ and $|\rho_+^{e,(k)} - \rho_-^{e,(k)}| \leqslant 2\,\kappa$, due to (32) and (56) we have the inequality

$$|\rho_\pm^{e,(k+1)}(t,x)| \leqslant B_{\max}(T) + T\,\sigma_{\max}(\kappa) \tag{58}$$

with $B_{\max}(T)$ as defined in (42) and $\sigma_{\max}(\kappa)$ as defined in (35). Due to (43), (58) implies

$$|\rho_\pm^{e,(k+1)}(t,x)| \leqslant u_{\max}.$$

Moreover, due to (58) we have the inequality

$$|\rho_+^{e,(k+1)}(t,x) - \rho_-^{e,(k+1)}(t,x)| \leqslant 2\,B_{\max}(T) + 2\,T\,\sigma_{\max}(\kappa) \leqslant 2\,\kappa, \tag{59}$$

where the last inequality follows from (43).

By induction this implies that for all $k \in \{1, 2, 3, ...\}$ we have $(\rho_+^{e,(k)},\, \rho_-^{e,(k)}) \in M_1(u_{\max})$ and $|\rho_+^{e,(k)} - \rho_-^{e,(k)}| \leqslant 2\,\kappa$. In particular, this implies that the fixed point iteration is well-defined, since in each iteration step Lemma 3 yields the necessary characteristic curves.

In the next step we look at the Lipschitz constants and show that there exists a nonincreasing sequence of Lipschitz constants for the functions $\rho_\pm^{e,(k)}$ generated by the fixed point iteration.

Step 2 (Uniform boundedness of the Lipschitz constants)

Now we consider the Lipschitz constants of $\Phi_\pm(\rho_+,\, \rho_-)$ for given common Lipschitz constants $\tilde L_M$ of ρ_+ and ρ_-. For this purpose we set

$$\Phi_\pm(\rho_+, \rho_-)(t,x) = f_1(t,x) + f_2(t,x)$$

with

$$\begin{aligned} f_1^e(t,x) &= \rho_\pm^e(t_\pm^{\mathcal{R}^e}(x,t),\, \xi_\pm^{\mathcal{R}^e}(t_\pm^{\mathcal{R}^e}(x,t),\, x,\, t)),\\ f_2^e(t,x) &= -\int_{t_\pm^{\mathcal{R}^e}(x,t)}^{t} \pm\,\sigma^e(\rho_+^e,\, \rho_-^e)\,\Big(\tau,\, \xi_\pm^{\mathcal{R}^e}(\tau,\, x,\, t)\Big)\, d\tau \end{aligned}$$

where $\mathcal{R}^e = (\rho_+^e,\, \rho_-^e)$. First we look at the Lipschitz constant of f_2.

For our analysis we also need the Lipschitz constants of $t_\pm^{\mathcal{R}^e}(x,t)$ with respect to x. According to (54) this is

$$L_B = \max_{e\in E} \frac{1}{\underline{\Lambda}^e(u_{\max})}\ \exp(2\,\Lambda^e(u_{\max})\,\tilde L_M\,T).$$

The Lipschitz constant of $\xi_\pm^{\mathcal{R}^e}(s,\, x,\, t)$ with respect to s is

$$L_s = \max_{e\in E} \Lambda^e(u_{\max})$$

and the Lipschitz constant of $\xi_\pm^{\mathcal{R}^e}(s,\, x,\, t)$ with respect to x is according to (53)

$$L_x = \max_{e\in E} \exp(2\,\tilde L_M\,T\,\Lambda^e(u_{\max})).$$

With the Lipschitz constant L_σ that satisfies (36) we have

$$
\begin{aligned}
&|f_2(t,x_1) - f_2(t,x_2)|\\
=&\left|\int_{t_\pm^{\mathcal{R}^e}(x_1,t)}^t \sigma^e(\rho_+^e,\, \rho_-^e)\, \Big(\tau,\, \xi_\pm^{\mathcal{R}^e}(\tau,\, x_1,\, t)\Big)\, d\tau - \int_{t_\pm^{\mathcal{R}^e}(x_2,t)}^t \sigma^e(\rho_+^e,\, \rho_-^e)\, \Big(\tau,\, \xi_\pm^{\mathcal{R}^e}(\tau,\, x_2,\, t)\Big)\, d\tau\right|\\
\leqslant&\left|\int_{t_\pm^{\mathcal{R}^e}(x_1,t)}^t \sigma^e(\rho_+^e,\, \rho_-^e)\, \Big(\tau,\, \xi_\pm^{\mathcal{R}^e}(\tau,\, x_1,\, t)\Big)\, d\tau - \int_{t_\pm^{\mathcal{R}^e}(x_1,t)}^t \sigma^e(\rho_+^e,\, \rho_-^e)\, \Big(\tau,\, \xi_\pm^{\mathcal{R}^e}(\tau,\, x_2,\, t)\Big)\, d\tau\right|\\
+&\left|\int_{t_\pm^{\mathcal{R}^e}(x_1,t)}^t \sigma^e(\rho_+^e,\, \rho_-^e)\, \Big(\tau,\, \xi_\pm^{\mathcal{R}^e}(\tau,\, x_2,\, t)\Big)\, d\tau - \int_{t_\pm^{\mathcal{R}^e}(x_2,t)}^t \sigma^e(\rho_+^e,\, \rho_-^e)\, \Big(\tau,\, \xi_\pm^{\mathcal{R}^e}(\tau,\, x_2,\, t)\Big)\, d\tau\right|\\
\leqslant&\int_0^t \left|\sigma^e(\rho_+^e,\, \rho_-^e)\, \Big(\tau,\, \xi_\pm^{\mathcal{R}^e}(\tau,\, x_1,\, t)\Big) - \sigma^e(\rho_+^e,\, \rho_-^e)\, \Big(\tau,\, \xi_\pm^{\mathcal{R}^e}(\tau,\, x_2,\, t)\Big)\right| d\tau\\
+&\left|\int_{t_\pm^{\mathcal{R}^e}(x_1,t)}^{t_\pm^{\mathcal{R}^e}(x_2,t)} \sigma^e(\rho_+^e,\, \rho_-^e)\, \Big(\tau,\, \xi_\pm^{\mathcal{R}^e}(\tau,\, x_2,\, t)\Big)\, d\tau\right|\\
\leqslant&\, T\, 2\, L_\sigma\, \tilde L_M\, L_x\, |x_1 - x_2| + \sigma_{\max}(\kappa)\, \left|t_\pm^{\mathcal{R}^e}(x_1,t) - t_\pm^{\mathcal{R}^e}(x_2,t)\right|\\
\leqslant&\, T\, 2\, L_\sigma\, \tilde L_M\, L_x\, |x_1 - x_2| + \sigma_{\max}(\kappa)\, L_B\, |x_1 - x_2|\\
=&\Big(T\, 2\, L_\sigma\, \tilde L_M\, L_x + \sigma_{\max}(\kappa)\, L_B\Big)\, |x_1 - x_2|\,.
\end{aligned}
$$

Thus f_2 has the Lipschitz constant

$$L_2 = 2\, T\, L_\sigma\, \tilde L_M\, L_x + \sigma_{\max}(\kappa)\, L_B.$$

Now we determine a Lipschitz constant for f_1. The definition of f_1 implies that for the evaluation of f_1 either given initial- and boundary data or data coming from (56) are evaluated at certain points. Let $x_1,\, x_2 \in [0,\, L^e]$ be given. We consider three cases:

Case 1.: $t_\pm^{\mathcal{R}^e}(x_1,t) = 0$ and $t_\pm^{\mathcal{R}^e}(x_2,t) = 0$. Then we have

$$
\begin{aligned}
&|f_1^e(t,x_1) - f_1^e(t,x_2)|\\
=&\left|\rho_\pm^e(t_\pm^{\mathcal{R}^e}(x_1,t),\, \xi_\pm^{\mathcal{R}^e}(t_\pm^{\mathcal{R}^e}(x_1,t),\, x_1,\, t)) - \rho_\pm^e(t_\pm^{\mathcal{R}^e}(x_2,t),\, \xi_\pm^{\mathcal{R}^e}(t_\pm^{\mathcal{R}^e}(x_2,t),\, x_2,\, t))\right|\\
=&\left|\rho_\pm^e(0,\, \xi_\pm^{\mathcal{R}^e}(0,\, x_1,\, t)) - \rho_\pm^e(0,\, \xi_\pm^{\mathcal{R}^e}(0,\, x_2,\, t))\right|\\
=&\left|y_\pm^e(\xi_\pm^{\mathcal{R}^e}(0,\, x_1,\, t)) - y_\pm^e(\xi_\pm^{\mathcal{R}^e}(0,\, x_2,\, t))\right|\\
\leqslant&\, \tilde L_R \left|\xi_\pm^{\mathcal{R}^e}(0,\, x_1,\, t) - \xi_\pm^{\mathcal{R}^e}(0,\, x_2,\, t)\right|\\
\leqslant&\, \tilde L_R\, L_x\, |x_1 - x_2|\,.
\end{aligned}
$$

Hence in Case 1 the function f_1 satisfies a Lipschitz inequality with the Lipschitz constant

$$\tilde L_R\, L_x.$$

Case 2.: $t_\pm^{\mathcal{R}^e}(x_1,t) > 0$ and $t_\pm^{\mathcal{R}^e}(x_2,t) > 0$. Then we have

$$\xi_\pm^{\mathcal{R}^e}(t_\pm^{\mathcal{R}^e}(x_1,t),\, x_1,\, t) = \xi_\pm^{\mathcal{R}^e}(t_\pm^{\mathcal{R}^e}(x_2,t),\, x_2,\, t) \tag{60}$$

since either $t_\pm^{\mathcal{R}^e}(x_1,t)$ and $t_\pm^{\mathcal{R}^e}(x_2,t)$ are both equal to zero or to L^e.

Case 2a.: If this boundary point of $[0, L^e]$ corresponds to a boundary node of the graph, this implies

$$\begin{aligned}
&|f_1^e(t,x_1) - f_1^e(t,x_2)| \\
=& \left|\rho_\pm^e(t_\pm^{\mathcal{R}^e}(x_1,t),\, \xi_\pm^{\mathcal{R}^e}(t_\pm^{\mathcal{R}^e}(x_1,t),\, x_1,\, t)) - \rho_\pm^e(t_\pm^{\mathcal{R}^e}(x_2,t),\, \xi_\pm^{\mathcal{R}^e}(t_\pm^{\mathcal{R}^e}(x_2,t),\, x_2,\, t))\right| \\
=& \left|u_\pm^e(t_\pm^{\mathcal{R}^e}(x_1,t)) - u_\pm^e(t_\pm^{\mathcal{R}^e}(x_2,t))\right| \\
\leqslant& \tilde{L}_R \ \left|t_\pm^{\mathcal{R}^e}(x_1,t) - t_\pm^{\mathcal{R}^e}(x_2,t)\right| \\
\leqslant& \tilde{L}_R\, L_B\ |x_1 - x_2|\,.
\end{aligned}$$

Case 2b.: If the boundary point of $[0, L^e]$ from (60) corresponds to an interior node $v \in V$ of the graph, we have

$$\begin{aligned}
&|f_1^e(t,x_1) - f_1^e(t,x_2)| \\
=& \left|\rho_\pm^e(t_\pm^{\mathcal{R}^e}(x_1,t),\, \xi_\pm^{\mathcal{R}^e}(t_\pm^{\mathcal{R}^e}(x_1,t),\, x_1,\, t)) - \rho_\pm^e(t_\pm^{\mathcal{R}^e}(x_2,t),\, \xi_\pm^{\mathcal{R}^e}(t_\pm^{\mathcal{R}^e}(x_2,t),\, x_2,\, t))\right| \\
=& \left|\Omega_v^e\, \rho_{in}^v(t_\pm^{\mathcal{R}^e}(x_1,t)) - \Omega_v^e\, \rho_{in}^v(t_\pm^{\mathcal{R}^e}(x_2,t))\right| \\
\leqslant& \left\|\rho_{in}^v(t_\pm^{\mathcal{R}^e}(x_1,t)) - \rho_{in}^v(t_\pm^{\mathcal{R}^e}(x_2,t))\right\|_\infty
\end{aligned}$$

where as usual $\|\cdot\|_\infty$ denotes the maximum-norm. The components of ρ_{in}^v that appear in the last line are in turn obtained by integrating along the characteristic curves $\xi_\pm^{\mathcal{R}^e}$. Due to (39), they can be followed back to the initial state, that is for $f \in E_0(v)$ the components of ρ_{in}^v have the form

$$\rho_\pm^f(t,\, x^f(v)) = y_\pm^f(\xi_\pm^{\mathcal{R}^f}(0,\, x,\, t)) \mp \int_0^t \sigma^f(\rho_+^f,\, \rho_-^f)\, \left(\tau,\, \xi_\pm^{\mathcal{R}^f}(\tau,\, x,\, t)\right)\, d\tau \tag{61}$$

without further reflections. Hence with the Lipschitz constant for f_2 and the Lipschitz constant from Case 1. we get the inequality

$$\begin{aligned}
&|\rho_\pm^f(t_\pm^{\mathcal{R}^e}(x_1,t),\, x^f(v)) - \rho_\pm^f(t_\pm^{\mathcal{R}^e}(x_2,t),\, x^f(v))| \\
\leqslant& (\tilde{L}_R\, L_x + L_2)\, |x_1 - x_2|.
\end{aligned}$$

This yields

$$\begin{aligned}
&|f_1^e(t,x_1) - f_1^e(t,x_2)| \\
\leqslant& (\tilde{L}_R\, L_x + L_2)\, |x_1 - x_2|.
\end{aligned}$$

Hence in Case 2 the function f_1 satisfies a Lipschitz inequality with the Lipschitz constant

$$\max\{\tilde{L}_R\, L_B,\, \tilde{L}_R\, L_x + L_2\}.$$

Case 3.: $t_\pm^{\mathcal{R}^e}(x_1,t) > 0$ and $t_\pm^{\mathcal{R}^e}(x_2,t) = 0$.

If $\xi_\pm^{\mathcal{R}^e}(t_\pm^{\mathcal{R}^e}(x_1,t),\, x_1,\, t) \in \{0,\, L^e\}$ corresponds to a boundary node of the graph, due to (40), (41) we have

$$
\begin{aligned}
& |f_1^e(t,x_1) - f_1^e(t,x_2)| \\
= & \left|\rho^e_\pm(t^{\mathcal{R}^e}_\pm(x_1,t),\, \xi^{\mathcal{R}^e}_\pm(t^{\mathcal{R}^e}_\pm(x_1,t),\, x_1,\, t)) - \rho^e_\pm(0,\, \xi^{\mathcal{R}^e}_\pm(0,\, x_2,\, t))\right| \\
= & \left|u^e_\pm(t^{\mathcal{R}^e}_\pm(x_1,t),\, \xi^{\mathcal{R}^e}_\pm(t^{\mathcal{R}^e}_\pm(x_1,t),\, x_1,\, t)) - y^e_\pm(\xi^{\mathcal{R}^e}_\pm(0,\, x_2,\, t))\right| \\
\leqslant & \tilde{L}_R \left(\left|t^{\mathcal{R}^e}_\pm(x_1,t)\right| + \left|\xi^{\mathcal{R}^e}_\pm(t^{\mathcal{R}^e}_\pm(x_1,t),\, x_1,\, t) - \xi^{\mathcal{R}^e}_\pm(t^{\mathcal{R}^e}_\pm(x_2,t),\, x_2,\, t)\right|\right) \\
= & \tilde{L}_R \left(\left|t^{\mathcal{R}^e}_\pm(x_1,t) - t^{\mathcal{R}^e}_\pm(x_2,t)\right| + \left|\xi^{\mathcal{R}^e}_\pm(t^{\mathcal{R}^e}_\pm(x_1,t),\, x_1,\, t) - \xi^{\mathcal{R}^e}_\pm(t^{\mathcal{R}^e}_\pm(x_2,t),\, x_2,\, t)\right|\right) \\
\leqslant & \tilde{L}_R\, L_B\ \ |x_1 - x_2| + \tilde{L}_R\, L_s\ \left|t^{\mathcal{R}^e}_\pm(x_1,t) - t^{\mathcal{R}^e}_\pm(x_2,t)\right| + \tilde{L}_R\, L_x\ |x_1 - x_2| \\
= & \tilde{L}_R\ [L_B\ + L_s\, L_B + L_x]\ |x_1 - x_2|\,.
\end{aligned}
$$

If $\xi^{\mathcal{R}^e}_\pm(t^{\mathcal{R}^e}_\pm(x_1,t),\, x_1,\, t) \in \{0,\, L^e\}$ corresponds to an interior node v of the graph, due to (40), (41) and the compatibility of the initial data with the node conditions we have

$$
\begin{aligned}
& |f_1^e(t,x_1) - f_1^e(t,x_2)| \\
= & \left|\rho^e_\pm(t^{\mathcal{R}^e}_\pm(x_1,t),\, \xi^{\mathcal{R}^e}_\pm(t^{\mathcal{R}^e}_\pm(x_1,t),\, x_1,\, t)) - \rho^e_\pm(t^{\mathcal{R}^e}_\pm(x_2,t),\, \xi^{\mathcal{R}^e}_\pm(t^{\mathcal{R}^e}_\pm(x_2,t),\, x_2,\, t))\right| \\
= & \left|\Omega^e_v\, \rho^v_{in}(t^{\mathcal{R}^e}_\pm(x_1,t)) - y^e_\pm(\xi^{\mathcal{R}^e}_\pm(t^{\mathcal{R}^e}_\pm(x_2,t),\, x_2,\, t))\right| \\
\leqslant & \left|\Omega^e_v\, \rho^v_{in}(t^{\mathcal{R}^e}_\pm(x_1,t)) - \Omega^e_v\, \rho^v_{in}(0)\right| + \left|\Omega^e_v\, \rho^v_{in}(0) - y^e_\pm(\xi^{\mathcal{R}^e}_\pm(0,\, x_2,\, t))\right| \\
\leqslant & \left\|\rho^v_{in}(t^{\mathcal{R}^e}_\pm(x_1,t)) - \rho^v_{in}(0)\right\|_\infty + \left|y^e_\pm(\xi^{\mathcal{R}^e}_\pm(0,\, x_1,\, t)) - y^e_\pm(\xi^{\mathcal{R}^e}_\pm(0,\, x_2,\, t))\right| \\
\leqslant & (\tilde{L}_R\, L_x + L_2)\, |x_1 - x_2| + \tilde{L}_R\, L_x\, |x_1 - x_2| \\
= & \left[2\, \tilde{L}_R\, L_x + L_2\right]\ |x_1 - x_2|\,.
\end{aligned}
$$

Here the upper bound for $\left\|\rho^v_{in}(t^{\mathcal{R}^e}_\pm(x_1,t)) - \rho^v_{in}(0)\right\|_\infty$ is obtained as in Case 2b.

Hence in Case 3 the function f_1 satisfies a Lipschitz inequality with the Lipschitz constant

$$\max\{\tilde{L}_R\ [L_B\ + L_s\, L_B + L_x]\,,\ 2\, \tilde{L}_R\, L_x + L_2\}.$$

Collecting the results from Case 1 to Case 3 yields the Lipschitz constant for f_1

$$L_1 = \tilde{L}_R\ [L_B\ + L_s\, L_B + 2\, L_x] + L_2.$$

This implies the inequality

$$
\begin{aligned}
& |\Phi_\pm(\rho_+, \rho_-)(t,x_1) - \Phi_\pm(\rho_+, \rho_-)(t,x_2)| \\
\leqslant & (L_1 + L_2)\ |x_1 - x_2| = L_\Phi\ |x_1 - x_2|
\end{aligned}
$$

with the Lipschitz constant

$$
\begin{aligned}
& L_\Phi = L_1 + L_2 \\
= {} & \tilde{L}_R\ [L_B\ + L_s\ L_B + 2\,L_x] + 2\,L_2 \\
= {} & \tilde{L}_R\ [L_B(1+L_s) + 2\,L_x] + 2\ \left[2\,T\,L_\sigma\,\tilde{L}_M\,L_x + \sigma_{\max}(\kappa)\,L_B\right] \\
= {} & \tilde{L}_R\ \left[\max_{e\in E} \frac{\mathrm{e}^{2\,\Lambda^e(u_{\max})\,\tilde{L}_M\,T}}{\underline{\Lambda}^e(u_{\max})} \left(1 + \max_{e\in E} \Lambda^e(u_{\max})\right) + 2\,\max_{e\in E} \mathrm{e}^{2\,\Lambda^e(u_{\max})\,\tilde{L}_M\,T}\right] \\
+ {} & 2\ \left[2\,T\,L_\sigma\,\tilde{L}_M\ \max_{e\in E} \mathrm{e}^{2\,\Lambda^e(u_{\max})\,\tilde{L}_M\,T} + \sigma_{\max}(\kappa)\ \max_{e\in E} \frac{\mathrm{e}^{2\,\Lambda^e(u_{\max})\,\tilde{L}_M\,T}}{\underline{\Lambda}^e(u_{\max})}\right].
\end{aligned}
$$

We have assumed that $\tilde{L}_R$, $T\,L_\sigma$ and $\sigma_{\max}(\kappa)$ are sufficiently small such that (44) holds. Hence we have the inequality

$$L_\Phi \leqslant \tilde{L}_M. \tag{62}$$

This shows that the Lipschitz constant with respect to x is not increased during the fixed point iteration.

With Step 1 of the proof this implies that $(\rho_+^{(k+1)},\, \rho_-^{(k+1)}) \in M_2(\tilde{L}_M)$. Hence we have

$$\Phi(M_2(\tilde{L}_M)) \subset M_2(\tilde{L}_M).$$

Step 3: Contractivity The next step is to show that Φ is a contraction. Let $(R_+^e,\, R_-^e)$, $(S_+^e,\, S_-^e) \in M_2(\tilde{L}_M)$. For $(t,x) \in [0,T] \times [0, L^e]$, the definition of $\Phi_\pm$ implies the inequality

$$\left|\Phi_\pm^e(R_+^e, R_-^e) - \Phi_\pm^e(S_+^e, S_-^e)\right|(t,x) \leqslant A^e + I^e,$$

with

$$
\begin{aligned}
A^e &= \left| R_\pm(t_\pm^{\mathcal{R}^e}(x,t),\, \xi_\pm^{\mathcal{R}^e}(t_\pm^{\mathcal{R}^e}(x,t),\, x,\, t)) - S_\pm(t_\pm^{\mathcal{S}^e}(x,t),\, \xi_\pm^{\mathcal{S}^e}(t_\pm^{\mathcal{S}^e}(x,t),\, x,\, t))\right|, \\
I^e &= \int_0^t \left|\sigma^e(R_+^e,\, R_-^e)\left(\tau,\, \xi_\pm^{\mathcal{R}^e}(\tau,\, x,\, t)\right) - \sigma^e(S_+^e,\, S_-^e)\left(\tau,\, \xi_\pm^{\mathcal{S}^e}(\tau,\, x,\, t)\right)\right|\, d\tau.
\end{aligned}
$$

We have

$$
\begin{aligned}
I^e \leqslant & \int_0^t L_\sigma\ \left|R_+^e\left(\tau,\, \xi_\pm^{\mathcal{R}^e}(\tau,\, x,\, t)\right) - S_+^e\left(\tau,\, \xi_\pm^{\mathcal{S}^e}(\tau,\, x,\, t)\right)\right|\, d\tau \\
& + \int_0^t L_\sigma\ \left|R_-^e\left(\tau,\, \xi_\pm^{\mathcal{R}^e}(\tau,\, x,\, t)\right) - S_-^e\left(\tau,\, \xi_\pm^{\mathcal{S}^e}(\tau,\, x,\, t)\right)\right|\, d\tau \\
& \leqslant \int_0^t L_\sigma\ \left|R_+^e\left(\tau,\, \xi_\pm^{\mathcal{R}^e}(\tau,\, x,\, t)\right) - S_+^e\left(\tau,\, \xi_\pm^{\mathcal{R}^e}(\tau,\, x,\, t)\right)\right|\, d\tau \\
& + \int_0^t L_\sigma\ \left|S_+^e\left(\tau,\, \xi_\pm^{\mathcal{R}^e}(\tau,\, x,\, t)\right) - S_+^e\left(\tau,\, \xi_\pm^{\mathcal{S}^e}(\tau,\, x,\, t)\right)\right|\, d\tau \\
& + \int_0^t L_\sigma\ \left|R_-^e\left(\tau,\, \xi_\pm^{\mathcal{R}^e}(\tau,\, x,\, t)\right) - S_-^e\left(\tau,\, \xi_\pm^{\mathcal{R}^e}(\tau,\, x,\, t)\right)\right|\, d\tau \\
& + \int_0^t L_\sigma\ \left|S_-^e\left(\tau,\, \xi_\pm^{\mathcal{R}^e}(\tau,\, x,\, t)\right) - S_-^e\left(\tau,\, \xi_\pm^{\mathcal{S}^e}(\tau,\, x,\, t)\right)\right|\, d\tau \\
& \leqslant 2\,T\,L_\sigma\ \|\mathcal{R}^e - \mathcal{S}^e\|_{C([0,T]\times[0,L^e])^2} + 2\,T\,L_\sigma\,\tilde{L}_M\ \left\|\xi_\pm^{\mathcal{R}^e} - \xi_\pm^{\mathcal{S}^e}\right\|_{C([0,T]\times[0,L])}.
\end{aligned}
$$

Due to (49) from Lemma (3) this implies the inequality

$$I^e \leqslant 2\,T\,L_\sigma\,\left(1+\tilde{L}_M\,T\,\Lambda^e(u_{\max})\,\exp(2\Lambda^e(u_{\max})\,\tilde{L}_M\,T)\right)\,\|\mathcal{R}^e-\mathcal{S}^e\|_{C([0,T]\times[0,L^e])^2}.$$

Now we look at the term A^e. Without loss of generality we assume that $t_\pm^{\mathcal{R}^e}(x,t) \leqslant t_\pm^{\mathcal{S}^e}(x,t)$. We consider three cases:

Case 1.: $t_\pm^{\mathcal{S}^e}(x,t)=0$ and $t_\pm^{\mathcal{R}^e}(x,t)=0$. Then we have

$$\begin{aligned}
A^e &= \left|R_\pm(t_\pm^{\mathcal{R}^e}(x,t),\,\xi_\pm^{\mathcal{R}^e}(t_\pm^{\mathcal{R}^e}(x,t),\,x,\,t)) - S_\pm(t_\pm^{\mathcal{S}^e}(x,t),\,\xi_\pm^{\mathcal{S}^e}(t_\pm^{\mathcal{S}^e}(x,t),\,x,\,t))\right|\\
&= \left|R_\pm(0,\,\xi_\pm^{\mathcal{R}^e}(0,\,x,\,t)) - S_\pm(0,\,\xi_\pm^{\mathcal{S}^e}(0,\,x,\,t))\right|\\
&= \left|y_\pm^e(\xi_\pm^{\mathcal{R}^e}(0,\,x,\,t)) - y_\pm^e(\xi_\pm^{\mathcal{S}^e}(0,\,x,\,t))\right|\\
&\leqslant \tilde{L}_R\,\left|\xi_\pm^{\mathcal{R}^e}(0,\,x,\,t)-\xi_\pm^{\mathcal{S}^e}(0,\,x,\,t)\right|\\
&\leqslant \tilde{L}_R\,T\,\Lambda^e(u_{\max})\,\exp(2\Lambda^e(u_{\max})\,\tilde{L}_M\,T)\,\|\mathcal{R}^e-\mathcal{S}^e\|_{C([0,T]\times[0,L^e])^2}.
\end{aligned}$$

Case 2.: $t_\pm^{\mathcal{S}^e}(x,t)>0$ and $t_\pm^{\mathcal{R}^e}(x,t)>0$. Then we have

$$\xi_\pm^{\mathcal{R}^e}(t_\pm^{\mathcal{R}^e}(x,t),\,x,\,t) = \xi_\pm^{\mathcal{S}^e}(t_\pm^{\mathcal{S}^e}(x,t),\,x,\,t) \tag{63}$$

since either $t_\pm^{\mathcal{R}^e}(x,t)$ and $t_\pm^{\mathcal{S}^e}(x,t)$ are both equal to zero or to L^e.

Case 2a.: If this boundary point of $[0,L^e]$ corresponds to a boundary node of the graph, we have

$$\begin{aligned}
A^e &= \left|R_\pm(t_\pm^{\mathcal{R}^e}(x,t),\,\xi_\pm^{\mathcal{R}^e}(t_\pm^{\mathcal{R}^e}(x,t),\,x,\,t)) - S_\pm(t_\pm^{\mathcal{S}^e}(x,t),\,\xi_\pm^{\mathcal{S}^e}(t_\pm^{\mathcal{S}^e}(x,t),\,x,\,t))\right|\\
&\leqslant \max\{\left|u_+^e(t_\pm^{\mathcal{R}^e}(x,t)) - u_+^e(t_\pm^{\mathcal{S}^e}(x,t))\right|,\,\left|u_-^e(t_\pm^{\mathcal{R}^e}(x,t)) - u_-^e(t_\pm^{\mathcal{S}^e}(x,t))\right|\}\\
&\leqslant \tilde{L}_R\,\left|t_\pm^{\mathcal{R}^e}(x,t)-t_\pm^{\mathcal{S}^e}(x,t)\right|\\
&\leqslant \tilde{L}_R\,\frac{T\,\Lambda^e(u_{\max})\,\exp(2\Lambda^e(u_{\max})\,\tilde{L}_M\,T)}{\underline{\Lambda}^e(u_{\max})}\,\|\mathcal{R}^e-\mathcal{S}^e\|_{C([0,T]\times[0,L^e])^2}.
\end{aligned}$$

Case 2b.: If the boundary point of $[0,L^e]$ from (63) corresponds to an interior node $v\in V$ of the graph, we have

$$\begin{aligned}
A^e &= \left|R_\pm(t_\pm^{\mathcal{R}^e}(x,t),\,\xi_\pm^{\mathcal{R}^e}(t_\pm^{\mathcal{R}^e}(x,t),\,x,\,t)) - S_\pm(t_\pm^{\mathcal{S}^e}(x,t),\,\xi_\pm^{\mathcal{S}^e}(t_\pm^{\mathcal{S}^e}(x,t),\,x,\,t))\right|\\
&= \left|\Omega_v^e\,R_{in}^v(t_\pm^{\mathcal{R}^e}(x,t)) - \Omega_v^e\,S_{in}^v(t_\pm^{\mathcal{S}^e}(x,t))\right|_\infty
\end{aligned}$$

The components of R_{in}^v and S_{in}^v that appear in the last line are in turn obtained by integrating along the characteristic curves $\xi_\pm^{\mathcal{R}^e}$. Due to (39), they can be followed back to the initial state, that is for $f\in E_0(v)$ the components of $R_{in}^v(t)$, $S_{in}^v(t)$, have the form (61)

$$\rho_\pm^f(t,\,x^f(v)) = y_\pm^f(\xi_\pm^{\mathcal{R}^f}(0,\,x,\,t)) \mp \int_0^t \sigma^f(\rho_+^f,\,\rho_-^f)\,\left(\tau,\,\xi_\pm^{\mathcal{R}^f}(\tau,\,x,\,t)\right)\,d\tau \tag{64}$$

without further reflections. Hence with the Lipschitz constant for I^e and the Lipschitz constant from Case 1. we get the inequality

$$A^e \leqslant L_{Case2b}\|R_{in}^v(t_\pm^{\mathcal{R}^e}(x,t)) - S_{in}^v(t_\pm^{\mathcal{S}^e}(x,t))\|_\infty \tag{65}$$

with

$$L_{Case2b} = \tilde{L}_R\, T\, \Lambda^e(u_{\max})\, \exp(2\Lambda^e(u_{\max})\, \tilde{L}_M\, T) + 2\, T\, L_\sigma \, \Big(1 + \tilde{L}_M\, T\, \Lambda^e(u_{\max})\, \exp(2\Lambda^e(u_{\max})\, \tilde{L}_M\, T)\Big).$$

Case 3.: $t_\pm^{\mathcal{S}^e}(x,t) > 0$ and $t_\pm^{\mathcal{R}^e}(x,t) = 0$. If $\xi_\pm^{\mathcal{R}^e}(t_\pm^{\mathcal{R}^e}(x,t),\, x,\, t) \in \{0,\, L^e\}$ corresponds to a boundary node v of the graph, we have

$$\begin{aligned}
A^e &= \Big|R_\pm(t_\pm^{\mathcal{R}^e}(x,t),\, \xi_\pm^{\mathcal{R}^e}(t_\pm^{\mathcal{R}^e}(x,t),\, x,\, t)) - S_\pm(t_\pm^{\mathcal{S}^e}(x,t),\, \xi_\pm^{\mathcal{S}^e}(t_\pm^{\mathcal{S}^e}(x,t),\, x,\, t))\Big| \\
&= \Big|R_\pm(0,\, \xi_\pm^{\mathcal{R}^e}(0,\, x,\, t)) - S_\pm(t_\pm^{\mathcal{S}^e}(x,t),\, \xi_\pm^{\mathcal{S}^e}(t_\pm^{\mathcal{S}^e}(x,t),\, x,\, t))\Big| \\
&\leqslant \tilde{L}_R\, \Big(\Big|t_\pm^{\mathcal{R}^e}(x,\, t) - t_\pm^{\mathcal{S}^e}(x,\, t)\Big| + \Big|\xi_\pm^{\mathcal{R}^e}(0,\, x,\, t) - \xi_\pm^{\mathcal{S}^e}(0,\, x,\, t)\Big|\Big) \\
&\leqslant \tilde{L}_R\, \left(1 + \frac{1}{\underline{\Lambda}^e(u_{\max})}\right) T\, \Lambda^e(u_{\max})\, \exp(2\Lambda^e(u_{\max})\, \tilde{L}_M\, T)\, \|\mathcal{R}^e - \mathcal{S}^e\|_{C([0,T]\times[0,L^e])^2}.
\end{aligned}$$

If $\xi_\pm^{\mathcal{R}^e}(t_\pm^{\mathcal{R}^e}(x,t),\, x,\, t) \in \{0,\, L^e\}$ corresponds to an interior node v of the graph, we have

$$\begin{aligned}
A^e &= \Big|R_\pm(t_\pm^{\mathcal{R}^e}(x,t),\, \xi_\pm^{\mathcal{R}^e}(t_\pm^{\mathcal{R}^e}(x,t),\, x,\, t)) - S_\pm(t_\pm^{\mathcal{S}^e}(x,t),\, \xi_\pm^{\mathcal{S}^e}(t_\pm^{\mathcal{S}^e}(x,t),\, x,\, t))\Big| \\
&\leqslant \Big|\Omega_v^e\, S_{in}^v(t_\pm^{\mathcal{S}^e}(x,t)) - \Omega_v^e\, S_{in}^v(0)\Big| + \Big|\Omega_v^e\, S_{in}^v(0) - R_\pm(t_\pm^{\mathcal{R}^e}(x,t),\, \xi_\pm^{\mathcal{R}^e}(t_\pm^{\mathcal{R}^e}(x,t),\, x,\, t))\Big| \\
&\leqslant \Big|S_{in}^v(t_\pm^{\mathcal{S}^e}(x,t)) - S_{in}^v(0)\Big|_\infty + \Big|y_\pm(x^e(v)) - y_\pm(\xi_\pm^{\mathcal{R}^e}(0,\, x,\, t))\Big| \\
&\leqslant \tilde{L}_R\, \Big|t_\pm^{\mathcal{R}^e}(x,\, t) - t_\pm^{\mathcal{S}^e}(x,\, t)\Big| + \tilde{L}_R\, \Big|\xi_\pm^{\mathcal{R}^e}(0,\, x,\, t) - \xi_\pm^{\mathcal{S}^e}(0,\, x,\, t)\Big| \\
&\leqslant \tilde{L}_R\, \left(1 + \frac{1}{\underline{\Lambda}^e(u_{\max})}\right) T\, \Lambda^e(u_{\max})\, \exp(2\Lambda^e(u_{\max})\, \tilde{L}_M\, T)\, \|\mathcal{R}^e - \mathcal{S}^e\|_{C([0,T]\times[0,L^e])^2}.
\end{aligned}$$

Collecting the results from Case 1 to Case 3 yields the Lipschitz constant L_A for A^e as

$$L_A = T\, \Lambda^e(u_{\max}) e^{2\Lambda^e(u_{\max})\, \tilde{L}_M\, T}\, \left(\tilde{L}_R + \frac{\tilde{L}_R}{\underline{\Lambda}^e(u_{\max})} + 2\, T\, L_\sigma\, \tilde{L}_M\right) + 2\, T\, L_\sigma. \tag{66}$$

With our results for I^e and A^e we get the Lipschitz inequality for $\Phi_\pm^e$

$$\begin{aligned}
&\big\|\Phi_\pm^e(R_+^e,\, R_-^e) - \Phi_\pm^e(S_+^e, S_-^e)\big\|_{C([0,\,T]\times[0,L])} \\
&\leqslant I^e + A^e \\
&\leqslant \max_{e\in E} \Big[2\, T\, L_\sigma\, \Big(2 + \tilde{L}_M\, T\, \Lambda^e(u_{\max})\, \exp(2\Lambda^e(u_{\max})\, \tilde{L}_M\, T)\Big) \\
&+ T\, \Lambda^e(u_{\max}) e^{2\Lambda^e(u_{\max})\, \tilde{L}_M\, T}\, \left(\tilde{L}_R + \frac{\tilde{L}_R}{\underline{\Lambda}^e(u_{\max})} + 2\, T\, L_\sigma\, \tilde{L}_M\right)\Big]\, \|\mathcal{R}^e - \mathcal{S}^e\|_{C([0,T]\times[0,L^e])^2}
\end{aligned}$$

This implies

$$\begin{aligned}
&\big\|\Phi_+^e(R_+^e, R_-^e) - \Phi_+^e(S_+^e, S_-^e)\big\|_{C([0,T]\times[0,L^e])} \\
&+ \big\|\Phi_-^e(R_+^e, R_-^e) - \Phi_-^e(S_+^e, S_-^e)\big\|_{C([0,T]\times[0,L^e])} \\
&\leqslant L_{kontr}\, \Big(\big\|R_+^e - S_+^e\big\|_{C([0,T]\times[0,L^e])} + \big\|R_-^e - S_-^e\big\|_{C([0,T]\times[0,L^e])}\Big)
\end{aligned}$$

with the contraction constant

$$L_{kontr} = \max_{e\in E}\Big[2\,T\,L_\sigma\,\Big(2+\tilde{L}_M\,T\,\Lambda^e(u_{\max})\,\exp(2\Lambda^e(u_{\max})\,\tilde{L}_M\,T)\Big)$$
$$+\,T\,\Lambda^e(u_{\max})\mathrm{e}^{2\Lambda^e(u_{\max})\,\tilde{L}_M\,T}\,\left(\tilde{L}_R+\frac{\tilde{L}_R}{\underline{\Lambda}^e(u_{\max})}+2\,T\,L_\sigma\,\tilde{L}_M\right)\Bigg].$$

Due to (45), we have $L_{kontr} < 1$. Thus we have shown that for sufficently small values of T, $\tilde{L}_R$ and θ the map $\Phi = (\Phi_+, \Phi_-)$ is a contraction. Hence Banach's fixed point theorem implies the existence of a unique fixed point of the map, which solves our quasilinear initial boundary value problem ($\mathbf{S}$) .

For $t \in [0,\, T]$, define

$$U(t) = \max\left\{B_{\max}(T),\ \max_{(s,\,x)\in[0,\,t]\times[0,\,L^e]}\big(|R_+^e(s,\,x)|+|R_-^e(s,\,x)|\big)\right\}.$$

Since the solution $R_\pm^e$ is a fixed point of $\Phi_\pm^e$, the definition of $\Phi_\pm^e$ and (33) imply the integral inequality

$$U(t)\leqslant U(0)+\int_0^t \frac{1}{2}\,\theta^e\,\sqrt{R_s^e\,T^e}\,\kappa\,U(\tau)\,d\tau$$

for all $t\in[0,\,T]$. Now we can apply Gronwall's Lemma (see for example [17]). Since $U(0)\leqslant 2B_{\max}(T)$ we obtain

$$U(s)\leqslant U(0)\,\exp\left(\frac{1}{2}\,\theta^e\,\sqrt{R_s^e\,T^e}\,\kappa\,T\right)\leqslant 2\,\exp\left(\frac{1}{2}\,\theta^e\,\sqrt{R_s^e\,T^e}\,\kappa\,T\right)\,B_{\max}(T).$$

Thus we have shown the a–priori bound (46).

The definition of the Lipschitz constant L_Φ from (62) implies that the functions R_+^e and R_-^e are Lipschitz continuous with the Lipschitz constant from the a–priori bound (47).

Thus we have proved Theorem 2. □

Remark 3. *Theorem 2 states that for horizontal pipes semi-global solutions exist locally around the subsonic stationary states with $q^e = 0$. Similarly the existence of solutions can be shown locally around stationary states of the system with $q^e \neq 0$. These stationary systems have been studied in detail in [18].*

5. Semi–global solutions. In order to obtain solutions that exist on a larger time interval, we use the a-priori inequality (46) that gives an upper bound for the values of $|R_+^e|$ and $|R_-^e|$ and (47) that gives an upper bound for the corresponding Lipschitz constant. Choose

$$T_0\leqslant\frac{1}{2}\,\min_{e\in E}\frac{L^e}{\Lambda^e(u_{\max})}$$

and $\tilde{L}_R^0$ and κ^0 sufficiently small such that there is a number $\tilde{L}_M$ such that (44) and (45) hold. Let

$$C_0 = 2\,\exp\left(\frac{1}{2}\,\theta^e\,\sqrt{R_s^e\,T^e}\,\kappa^0\,T_0\right),$$

For $i\in\{2,3,...,N-1\}$, $\tilde{L}_r\leqslant\tilde{L}_R^0$ and $\kappa\leqslant\kappa_0$ let $C_L(i)$ be defined inductively as

$$C_L^1(\kappa,\,\tilde{L}_R)=\max_{e\in E}\frac{\mathrm{e}^{2\,\Lambda^e(u_{\max})\,\tilde{L}_M\,T_0}}{\underline{\Lambda}^e(u_{\max})}\left[\tilde{L}_R\left(1+\max_{e\in E}\Lambda^e(u_{\max})\right)+2\,\sigma_{\max}(\kappa)\right]\qquad(67)$$

$$+ \max_{e\in E} \mathrm{e}^{2\,\Lambda^e(u_{\max})\,\tilde L_M\,T_0}\,\left(2\,\tilde L_R + 4\,T_0\,L_\sigma\,\tilde L_M\right),$$

(68)

$$C_L^i(\kappa,\,\tilde L_R) = \max_{e\in E} \frac{\mathrm{e}^{2\,\Lambda^e(u_{\max})\,\tilde L_M\,T_0}}{\underline{\Lambda}^e(u_{\max})}\left[\tilde C_L^{i-1}(\kappa,\,\tilde L_R)\left(1+\max_{e\in E}\Lambda^e(u_{\max})\right) + 2\,\sigma_{\max}(\kappa)\right]$$

$$+ \max_{e\in E} \mathrm{e}^{2\,\Lambda^e(u_{\max})\,\tilde L_M\,T_0}\,\left(2\,C_L^{i-1}(\kappa,\,\tilde L_R) + 4\,T_0\,L_\sigma\,\tilde L_M\right).$$

Note that for all $j \in \{1, 2, ..., N-1\}$ we have

$$\lim_{\tilde L_R\to 0+,\,\kappa\to 0+} C_L^j(\kappa,\,\tilde L_R) = 0$$

since for L_σ as in (37) we have $\lim_{\kappa\to 0+} L_\sigma = 0$. Now let an arbitrary large time $T > 0$ be given. We choose a natural number $N \in \{1, 2, 3, ...\}$ such that $N\,T_0 \geqslant T$. Choose $\varepsilon > 0$ such that the assumptions of Theorem 2 hold for the time interval $[0,\,T_0]$ if $B_{\max}(T_0) \leqslant \varepsilon$. Due to (43) this requires

$$\varepsilon + \frac{1}{2}\,T_0\,\theta^e\,\sqrt{R_s^e\,T^e}\,\kappa_0^2 \leqslant \min\{u_{\max},\,\kappa_0\}. \tag{69}$$

By further decreasing $B_{\max}(T)$, we can make the bound so small that

$$B_{\max}(T) \leqslant \frac{1}{C_0^N}\varepsilon$$

Moreover by further decreasing $\tilde L_R$ and κ, we can make them so small that

$$C_L^{N-1}(\kappa,\,\tilde L_R) \leqslant \tilde L_R^0.$$

Then (46) implies that at the time T_0 the solution satisfies the inequality

$$\left|R_+^e(T_0,\,x)\right| + \left|R_-^e(T_0,\,x)\right| \leqslant B_{\max}(T)\;C_0 \leqslant \varepsilon.$$

Moreover, (47) implies that the functions $R_+^e(T_0,\,\cdot)$ and $R_-^e(T_0,\,\cdot)$ are Lipschitz continuous with the Lipschitz constant

$$C_L^1(\kappa,\,\tilde L_R) \leqslant \tilde L_R^0.$$

Thus we can apply Theorem 2 again to obtain the solution on the next time interval $[T_0,\; 2\,T_0]$. Since for all $i \in \{1, 2, ..., N-1\}$ we have

$$\left|R_+^e(i\,T_0,\,x)\right| + \left|R_-^e(i\,T_0,\,x)\right| \leqslant B_{\max}(T)\;C_0^i \leqslant \varepsilon$$

and the functions $R_+^e(i\,T_0,\,\cdot)$ and $R_-^e(i\,T_0,\,\cdot)$ are Lipschitz continuous with the Lipschitz constant

$$C_L^i(\kappa,\,\tilde L_R) \leqslant \tilde L_R^0$$

we can proceed inductively. By applying Theorem 2 N times we obtain the solution on the whole time interval $[0,\,T]$ if the boundary data, the initial data and their Lipschitz constants are sufficiently small. Thus we obtain the existence of semi–global solutions, that is for all times $T > 0$ the solution exists on $[0,\,T]$ if the $W^{1,\infty}$ norms of the initial data and the boundary controls are sufficiently small and compatible.

With our arguments we have proved the following theorem.

THEOREM 4. *Assume that for all $e \in E$, $s^e_{lope} = 0$ that is we consider a network of horizontal pipes. Assume that T^e and α^e are independent of $e \in E$.*

Let $T > 0$ be given. There exist numbers $\varepsilon(T) > 0$, $C_T > 0$ such that

$$C_T\,\varepsilon(T) < 1 - |\alpha^e|\,\exp(C_T\,\varepsilon(T)) \tag{70}$$

and for initial data $y^e_\pm$ and boundary data $u^e_\pm$ that satisfy the C^0-compatibility conditions and

$$|y^e_\pm(x)| \leqslant \varepsilon(T),\ |u^e_\pm(t)| \leqslant \varepsilon(T)$$

and are Lipschitz continuous with a Lipschitz constant that is less than or equal to $\varepsilon(T)$, System **(S)** *has a solution on $[0,\,T]$ that satisfies the a priori bound*

$$|R^e_+(t,\,x)| + |R^e_-(t,\,x)| \leqslant C_T\,\varepsilon(T)$$

and the solution is is Lipschitz continuous with a Lipschitz constant that is less than or equal to $C_T\,\varepsilon(T)$. Moreover, for the eigenvalues we have the bounds

$$\lambda^e_+ \geqslant \underline{\lambda}^e_+ := \sqrt{R^e_s\,T^e}\,[1 - |\alpha^e|\,\exp(C_T\,\varepsilon(T)) - C_T\,\varepsilon(T)]\,, \tag{71}$$

$$\lambda^e_- \leqslant \overline{\lambda}^e_- := \sqrt{R^e_s\,T^e}\,[-1 + |\alpha^e|\,\exp(C_T\,\varepsilon(T)) + C_T\,\varepsilon(T)]\,. \tag{72}$$

As in (18), inequality (70) implies that the state remains subsonic. The bounds for the eigenvalues are useful to estimate the time that the characteristic curves need to travel through the system until they reach a boundary node.

5.1. A maximum principle. In this Section we show that in addition to the a priori bound that we have already presented, due to the special form of the source term the solution satisfies a maximum principle in terms of the Riemann invariants.

We call the Riemann invariant R^e_+ incoming at the boundary $x = 0$ and outgoing at $x = L_e$ and the Riemann invariant R^e_- incoming at the boundary $x = L_e$ and outgoing at $x = 0$.

THEOREM 5. *Assume that for all $e \in E$, $x \in [0, L^e]$ the inequality $\theta^e(x) > 0$ holds. Consider an initial boundary value problem on a network $G = (V, E)$, where the node conditions of the Section 4 are used at the inner nodes with $u^v = 0$. Let $(R^e_+, R^e_-)_{e\in E}$ be a subsonic Lipschitz continuous solution on the time interval $[0, T]$. Then on any edge $e \in E$ the maximum*

$$R^e_{max} := \max_{t\in[0,T],x\in[0,L_e]} \max\left\{|R^e_+(t,x)|, |R^e_-(t,x)|\right\} \tag{73}$$

is attained by $|R^e_+(t,x)|$ on its inflow boundary $(\{0\}\times[0,L_e])\cup([0,T)\times\{0\})$ or by $|R^e_-(t,x)|$ on its inflow boundary $(\{0\}\times[0,L_e])\cup([0,T)\times\{L_e\})$.

Finally the global maximum on the network

$$R_{max} := \max_{e\in E}\ \max_{t\in[0,T],x\in[0,L_e]} \max\{|R^e_+(t,x)|, |R^e_-(t,x)|\} \tag{74}$$

is attained at the initial condition or at a boundary node by an incoming Riemann invariant (and hence by the boundary data).

Proof. In the proof we use the fact that along the characteristic curves the Lipschitz continuous solutions are differentiable.

Suppose that (73) is not attained by an incoming Riemann invariant on its inflow boundary.

Consider first the case that it is attained only in the interior w.l.o.g. by $R^e_+(\bar t,\bar x) > 0$ at some point $(\bar t,\bar x) \in (0,T)\times(0,L_e)$. Let $(t, \xi_\pm^{\mathcal{R}^e}(t,\bar x,\bar t))$ be the characteristics through $(\bar t,\bar x)$, which we abbreviate by $(t,\xi_\pm^{\mathcal{R}^e}(t))$. Then we have

$$\frac{d}{dt} R^e_+(t,\xi_+^{\mathcal{R}^e}(t))|_{t=\bar t} = 0.$$

Thus (20) implies $R^e_-(\bar t,\bar x) = R^e_+(\bar t,\bar x)$. Hence, also $R^e_-(\bar t,\bar x)$ attains the maximum (73).

There is a time $s\in[0,\bar t)$ at which $(s,\xi_+^{\mathcal{R}^e}(s))$ meets the inflow boundary $(\{0\}\times[0,L_e])\cup[0,T)\times\{0\}$ and by assumption $R^e_+(s,\xi_+^{\mathcal{R}^e}(s)) < R^e_+(\bar t,\bar x) = R^e_-(\bar t,\bar x) = R^e_{max}$ holds. Now (20) implies that the function $t\mapsto R^e_+(t,\xi_+^{\mathcal{R}^e}(t))$ satisfies the ordinary differential equation

$$\frac{d}{dt} R^e_+(t,\xi_+^{\mathcal{R}^e}(t)) = -\frac{1}{8}\theta^e\sqrt{R^e_sT^e}(|R^e_+ - R^e_-|(R^e_+ - R^e_-))(t,\xi_+^{\mathcal{R}^e}(t)), \quad t\in(s,\bar t). \tag{75}$$

Define the auxiliary function $t\mapsto \bar R^e_+(t)$ as solution of the initial value problem

$$\begin{aligned} &\frac{d}{dt}\bar R^e_+(t) = -\frac{1}{8}\theta^e\sqrt{R^e_sT^e}(|\bar R^e_+ - R^e_{\max}|(\bar R^e_+ - R^e_{\max}))(t), \quad t\in(s,\bar t),\\ &\bar R^e_+(s) = R^e_+(s,\xi_+^{\mathcal{R}^e}(s)). \end{aligned} \tag{76}$$

Since we have supposed that (73) is not attained by an incoming Riemann invariant on its inflow boundary, we have $\bar R^e_+(s) < R^e_{\max}$. Hence (76) has a strictly increasing solution that satisfies $\bar R^e_+(t) < R^e_{\max}$ for all $t\in[s,\bar t]$, namely

$$\bar R^e_+(t) = R^e_+(s,\xi_+^{\mathcal{R}^e}(s)) + \frac{\frac{1}{8}\theta^e\sqrt{R^e_sT^e}(R^e_{\max} - R^e_+(s,\xi_+^{\mathcal{R}^e}(s)))^2(t-s)}{\frac{1}{8}\theta^e\sqrt{R^e_sT^e}(R^e_{\max} - R^e_+(s,\xi_+^{\mathcal{R}^e}(s))(t-s)+1}.$$

Since $R^e_-(t,\xi_+^{\mathcal{R}^e}(t)) \le R^e_{\max}$, the right hand side of (76) majorizes the right hand side of of (75) and thus for all $t\in[s,\bar t]$ we have

$$R^e_+(t,\xi_+^{\mathcal{R}^e}(t)) \le \bar R^e_+(t) < R^e_{\max}.$$

This contradicts the supposition $R^e_+(\bar t,\xi_+^{\mathcal{R}^e}(\bar t)) = R^e_{\max}$.

If the maximum (73) is attained only in the interior by $R^e_-(\bar t,\bar x) > 0$ then as above $R^e_-(\bar t,\bar x) = R^e_+(\bar t,\bar x) = R^e_{\max}$ and also $R^e_+(\bar t,\bar x)$ attains the maximum value. Hence we can proceed exactly as above.

If the maximum (73) is attained only in the interior by $-R^e_+(\bar t,\bar x) = R^e_{\max}$ then we can argue as above by considering $-R^e_+$ and $-R^e_-$ instead of R^e_+ and R^e_-.

Now assume that the maximum (73) is attained by $R^e_+(\bar t,\bar x) > 0$ on the outflow boundary, i.e., $(\bar t,\bar x)\in(\{T\}\times[0,L_e])\cup[0,T)\times\{L_e\}$, but (73) is not attained by an incoming Riemann invariant R^e_+ or R^e_- on its inflow boundary. Let $(t,\xi_\pm^{\mathcal{R}^e}(t,\bar x,\bar t))$ be the characteristic curve through $(\bar t,\bar x)$, which we abbreviate by $(t,\xi_\pm^{\mathcal{R}^e}(t))$. Then we have

$$\frac{d}{dt} R^e_+(t,\xi_+^{\mathcal{R}^e}(t))|_{t=\bar t} \ge 0. \tag{77}$$

Now if $\bar t = T$ then we must have $R^e_+(\bar t,\bar x) = R^e_-(\bar t,\bar x) = R^e_{max}$, since otherwise $\frac{d}{dt}R^e_+(t,\xi_+^{\mathcal{R}^e}(t))|_{t=\bar t} < 0$ by (20), which is a contradiction to (77). But in the case $R^e_+(\bar t,\bar x) = R^e_-(\bar t,\bar x) = R^e_{max}$ we can derive a contradiction as for interior maxima.

Thus we have $\bar{x} = L_e$, $\bar{t} < T$ and $R_-^e(\bar{t}, \bar{x}) < R_+^e(\bar{t}, \bar{x})$, since otherwise the maximum (73) would also be attained by $R_-^e(\bar{t}, \bar{x})$ at its inflow boundary. Hence, we have again $\frac{d}{dt} R_+^e(t, \xi_+^{\mathcal{R}^e}(t))|_{t=\bar{t}} < 0$ by (20), which is again a contradiction to (77).

As above, the cases that (73) is attained as boundary maximum of outgoing Riemann invariants $R_-^e(\bar{t}, \bar{x})$, $-R_+^e(\bar{t}, \bar{x})$ or $-R_-^e(\bar{t}, \bar{x})$, respectively, at their outflow boundaries can be treated similar.

We know already that the global maximum (74) can only be attained by an incoming Riemann invariant at its inflow boundary. Assume that the global maximum (74) is attained at some interior node v. We have shown that there must then exist an edge $e \in E$ containing v such that (74) is attained in $(\bar{t}, x^e(v))$ at some time $\bar{t} < T$ by the absolut value of an incoming Riemann invariant, for which $(\bar{t}, x^e(v))$ is on the inflow boundary. Using (28), the boundary data R_{out}^v of the incoming Riemann invariants satisfy

$$\|R_{out}^v\|_\infty \leq \|\Omega^v\|_\infty \|R_{in}^v\|_\infty.$$

By (32) we know already that $\|\Omega^v\|_\infty \leq 1$, but for interior nodes at least two edges are incident to v and thus (31) shows that $\|\Omega^v\|_\infty < 1$. Hence, $\|R_{out}^v\|_\infty < \|R_{in}^v\|_\infty$ and thus there must exist an edge $f \in E_0(v)$ and an outgoing Riemann invariant $R_+^f(\bar{t}, x^f(v))$ or $R_-^f(\bar{t}, x^f(v))$ with $|R_\pm^f(\bar{t}, x^f(v))| > R_{max}$. This is impossible, since R_{max} is by (74) the global maximum. ∎

The maximum principle stated in Theorem 5 is useful for control problems with L^∞ state constraints. As an example consider an upper bound $p_{\max} > 1$ for the pressure that yields the constraint $p^e \leqslant p_{\max}$. Due to (11) and the monotonicity of the logarithm this is equivalent to

$$R_+^e + R_-^e = 2\ln(p^e) \leqslant 2\ \ln(p_{\max}). \tag{78}$$

Thus we have an upper bound for the sum of the Riemann invariants. The maximum principle states that the Riemann invariants attain the maximum values at the initial or boundary values. This allows to relax the state constraint: If

$$\max_{e \in E} \max_{t \in [0,T], x \in [0, L_e]} \max\{|R_+^e(t,x)|, |R_-^e(t,x)|\} \leqslant \ln(p_{\max}), \tag{79}$$

then (78) holds. Due to the maximum principle, in order to verify wether (79) holds, it suffices to check wether it is satisfied for the initial and boundary values.

6. Exact Controllability to stationary states. Based upon the existence result for the semi-global solutions of System **(S)** we study the exact boundary controllability of the system. To prove the exact controllability, we have to assume that the graph is tree–shaped. This is a standard assumptions that has also been used in the studies of exact controllability on channel networks, see [20]. Note that in contrast to the system in [20], if $\theta^e(x) > 0$ due to the nonzero source term System **(S)** is not a system of conservation laws but a system of balance laws. Therefore the stationary states of System **(S)** are in general not constant. A detailed analysis of the stationary states of System **(S)** on general networks (possibly including cycles) is given in [18].

In [20], piecewise classical solutions are considered whereas here we consider solutions that are only Lipschitz–continuous.

We show that if for all $e \in E$, $x \in [0, L^e]$ the equation $\theta^e(x) = 0$ holds, for a sufficiently small stationary state $S_\pm^e(x)$ and initial states that satisfy the assumptions of Theorem 4, there exist a time $T > 0$ and boundary controls such that at the time T, the generated system state satisfies the end conditions

$$R_\pm^e(T,\ x) = S_\pm^e(x). \tag{80}$$

Theorem 6. *Assume that $G = (V, E)$ is a tree and that for all $e \in E$, $x \in [0, L^e]$ the equation $\theta^e(x) = 0$ holds. Let $u_{\max} > 0$ be given such that (18) holds. Let*

$$T^* > \sum_{e\in E} \frac{L^e}{\sqrt{R_s^e\, T^e}} \frac{1}{1 - |\alpha^e|\, \exp(u_{\max}) - u_{\max}}. \tag{81}$$

By further reducing $\varepsilon(T^*)$ from Theorem 4, we can assume that

$$\varepsilon(T^*) \leqslant \frac{u_{\max}}{C_{T^*}}.$$

Then we have $\varepsilon(T^*)\, C_{T^*} \leqslant u_{\max}$, and thus

$$T^* > \sum_{e\in E} \frac{L^e}{|\underline{\lambda}_-^e|}. \tag{82}$$

Let a stationary state $(S_+^e(x),\, S_-^e(x))$ be given such that

$$\|S_\pm^e(x)\|_{C^1(0,\, L^e)} \leqslant \varepsilon(T^*)$$

for all $e \in E$. Assume that for all $t > 0$ we have $u_+^e(t) = S^e(0)$ and $u_-^e(t) = S^e(L^e)$. Assume that the initial state $(y_+^e,\, y_-^e)$ is C^0–compatible with $u_\pm^e$. If

$$|y_\pm^e(x)| \leqslant \varepsilon(T^*)$$

and the functions $y_\pm^e$ are Lipschitz continuous with a Lipschitz constant that is less than or equal to $\varepsilon(T^*)$, then the generated state satisfies the end conditions

$$R_\pm^e(T^*,\, x) = S_\pm^e(x). \tag{83}$$

Hence it can be continued as a stationary state for all $t > 0$ with the corresponding constant boundary controls.

Proof of Theorem 6: A characteristic curve that corresponds to λ_+^e needs at most the time $L^e/\overline{\lambda}_+^e$ to cross the edge e from the end $x = 0$ to the end $x = L^e$. A characteristic curve that corresponds to λ_-^e needs at most the time $L^e/|\underline{\lambda}_-^e|$ to go through the edge e from the end $x = L^e$ to the end $x = 0$.

Thus for states $(R_+^e,\, R_-^e)$ with $|R_\pm^e| \leqslant u_{\max}$ for all $e \in E$, due to (81) the characteristic curves through a point $(T^*,\, x)$ with $x \in L^e$, can be followed backwards in time until they reach a boundary node $v \in V$. Note that in following back through the nodes here means that at the nodes $v \in V$ that are not boundary nodes the characteristic curves branch. For a characteristic curve that corresponds to a component of R_{out}^v all the characteristic curves that correspond to components of R_{in}^v are followed backwards. Our assumption that G is a tree implies that during this process the characteristic curves cannot be trapped in cycles, since we can pass each node of the graph only once. By this construction, finally each of the curves ends a a boundary node $v \in V$ at a time $t > 0$. In the same way, the characteristic curves through a point $(T^*,\, x)$ with $x \in L^e$, can be followed forwards in time until they reach a boundary node.

Hence the solution at the time T^* is completely determined by the boundary data that contains the values for the stationary state and not influenced by the initial state. Therefore, (83) holds. □

Remark 4. *The case $\theta^e > 0$ will be the subject of future studies. We expect that also in this case it is possible to prove the exact controllability, similar as in [16].*

7. Conclusion. In this paper, we have shown the existence of Lipschitz continuous solutions for the isothermal Euler equations on networks. The regularity assumptions for these solutions are less restrictive than for classical solutions, but on the other hand, the solution is still determined by characteristic curves that are continuously differentiable. For these solutions we have shown a maximum principle in terms of the Riemann invariants and the local exact boundary controllabilty to stationary states on tree–shaped graphs if the steering time is sufficiently large.

Since the operation of gas pipeline networks is often based upon steady states (see for example [22]), this result is of interest since it implies that after finite time the system can indeed be controlled to such a stationary state.

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