The Subdivision of Large Simplicial Cones in Normaliz

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joint with
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& Christoph Söger

UNIVERSITÄT OSNABRÜCK

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- Open source software (GPL)
- written in C++ (using Boost and GMP/MPIR)
- parallelized with OpenMP
- runs under Linux, MacOs and MS Windows
- C++ library libnormaliz
- GUI interface jNormaliz

Version 3.1 just released!

http://www.math.uos.de/normaliz

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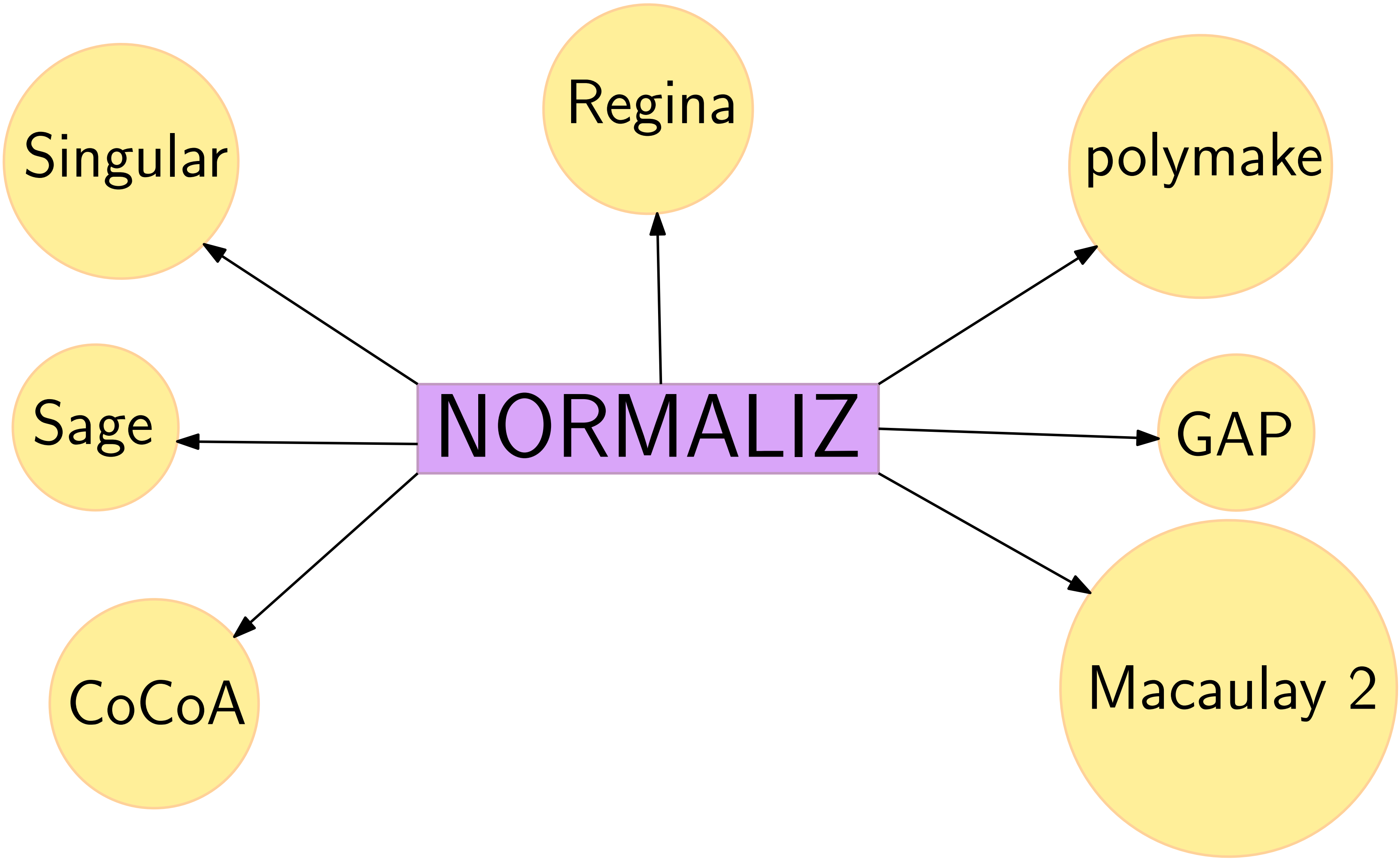

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Singular
polymake
Sage
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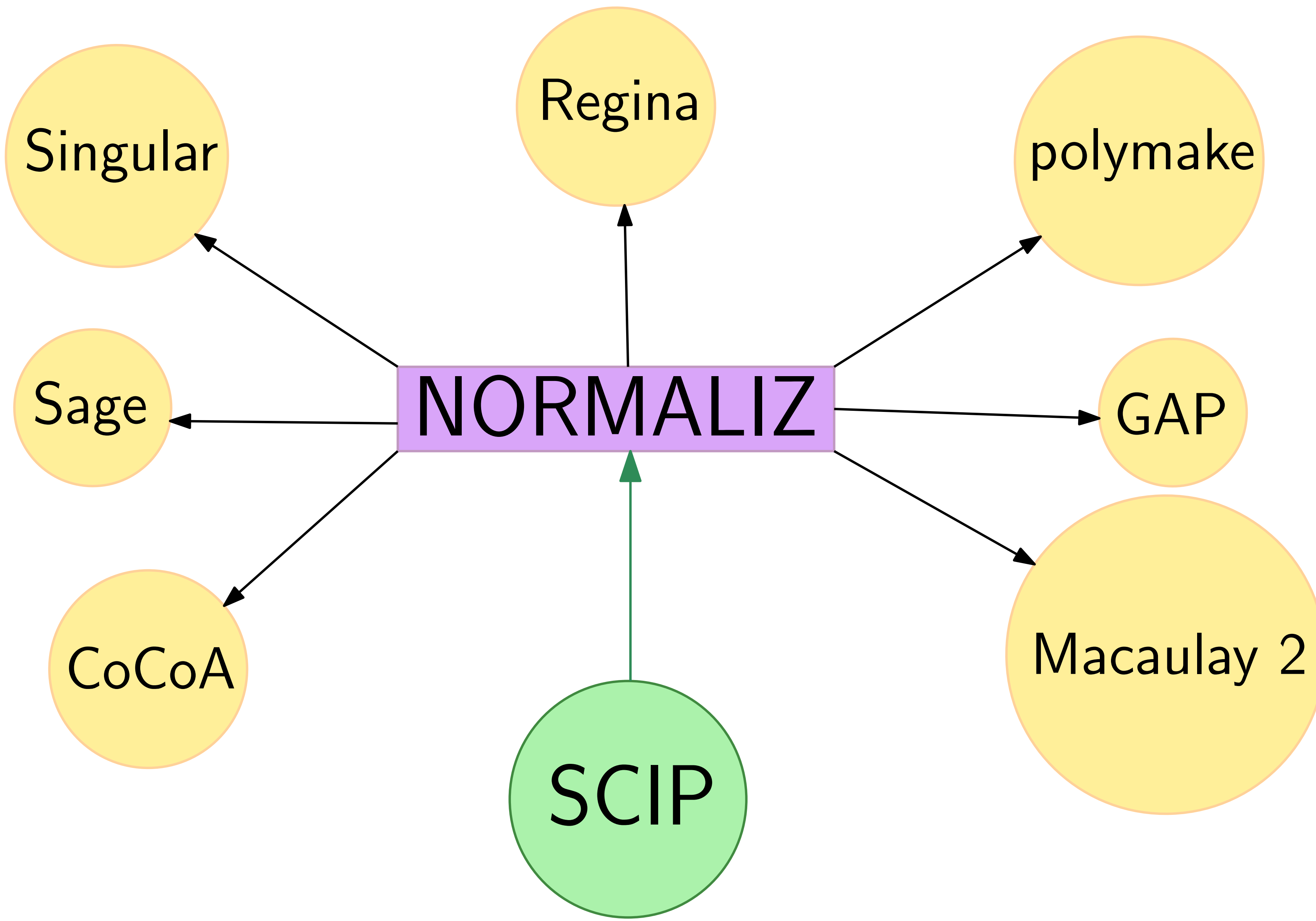
CoCoA
Macaulay 2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Rational Cones

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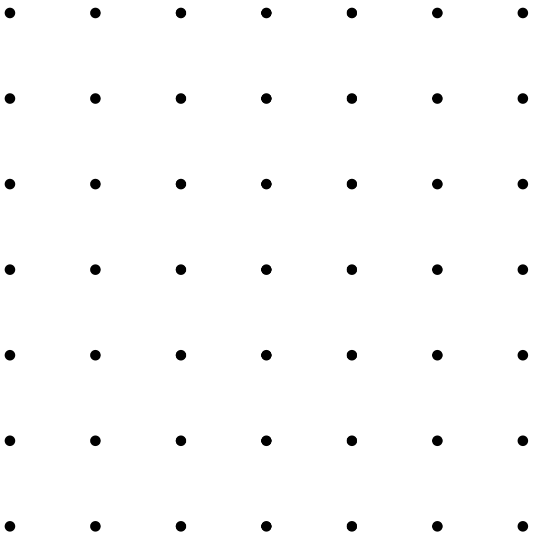

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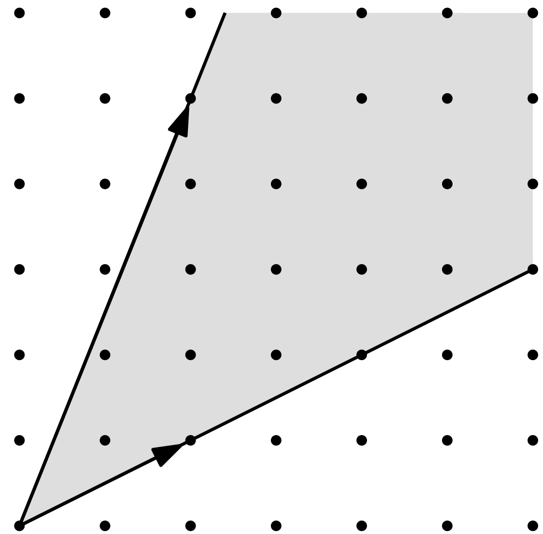
C . . . a (rational polyhedral) cone

$$\begin{aligned} C &= \operatorname{cone}(x_1, \ldots, x_n) \subset \mathbb{R}^d \\ &= \{a_1 x_1 + \cdots + a_n x_n \mid a_1, \ldots, a_n \in \mathbb{R}_+\} \\ &= \{x \in \mathbb{R}^n \mid Ax \geq 0\} \end{aligned}$$

with a generating system $x_1, \ldots, x_n \in \mathbb{Z}^d$.



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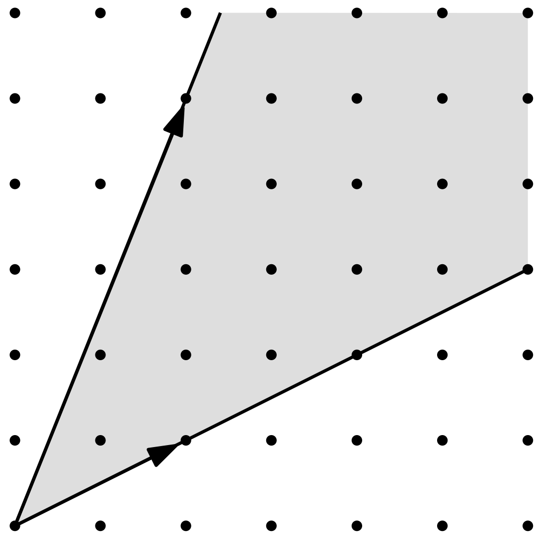

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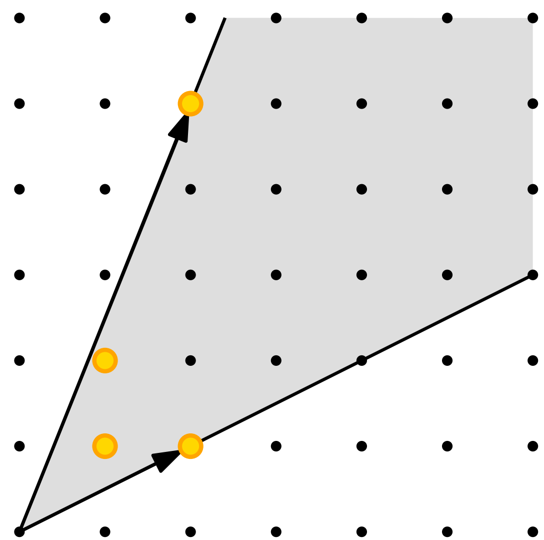

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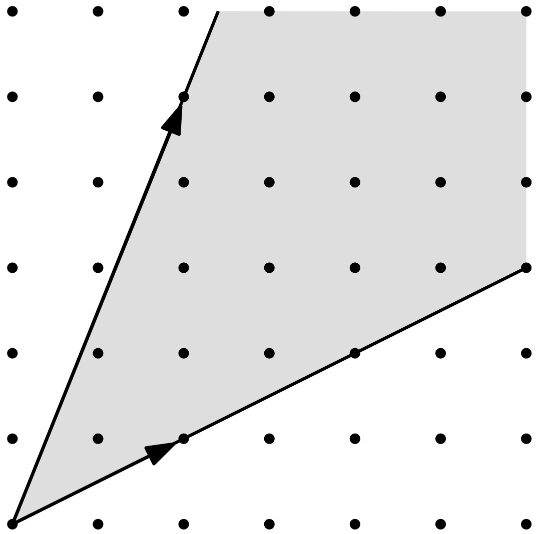
Theorem [Gordan's Lemma]

Let $C \subset \mathbb{R}^d$ be the cone generated by $x_1, \ldots, x_n \in \mathbb{Z}^d$. Then $C \cap L$ is an affine monoid M, i.e. a finitely generated submonoid of $\mathbb{Z}^d$.

The Tasks of Normaliz: Hilbert Basis

Assume C pointed: $x, -x \in C \Rightarrow x = 0$.



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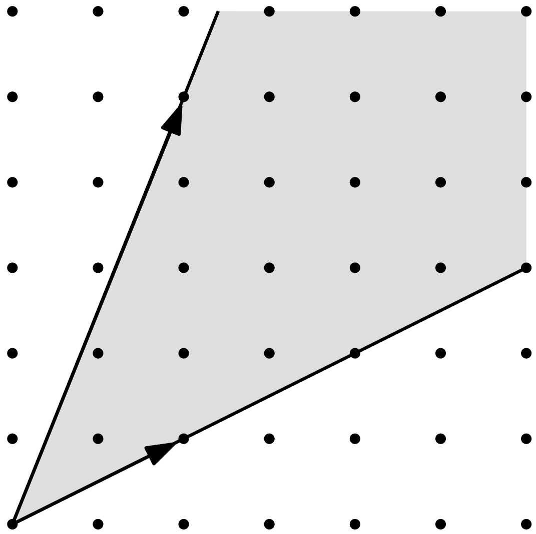
$x \in M = C \cap L, x \neq 0$ is irreducible:

$$x = y + z \;\Rightarrow\; y = 0 \text{ or } z = 0.$$



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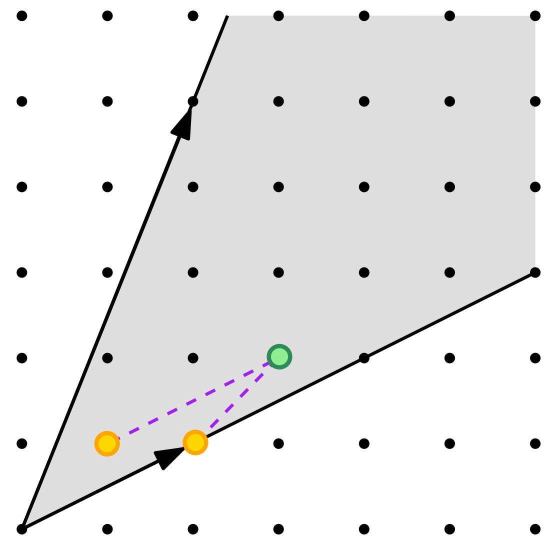

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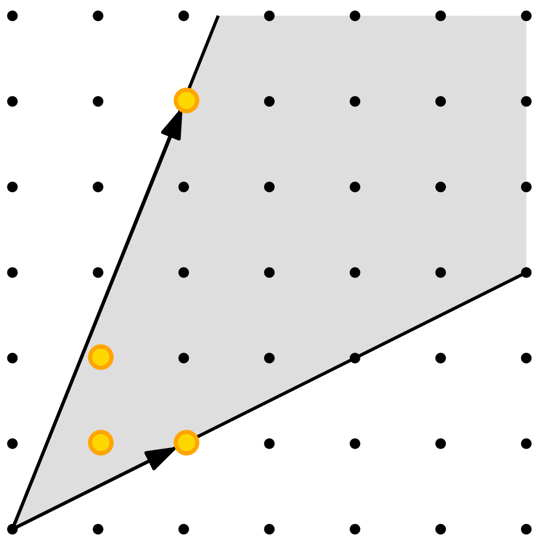

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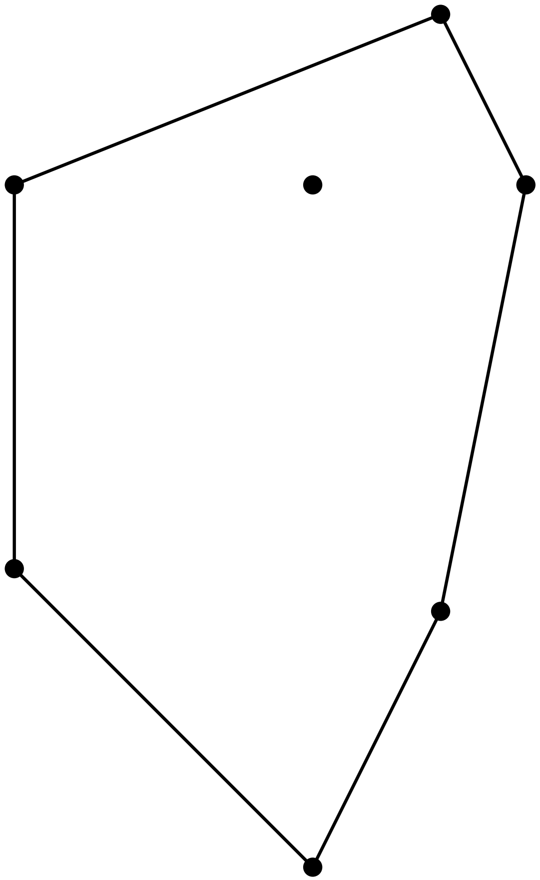
THEOREM [Hilbert's Basis Theorem]

There are only finitely many irreducible elements in $C \cap L$ and they form the unique minimal system of generators, the Hilbert Basis.

Normaliz Algorithm



cross section

In the Normaliz algorithm:

- Preparatory coordinate transformation, s.t. the cone is full dimensional and $L = \mathbb{Z}^d$.

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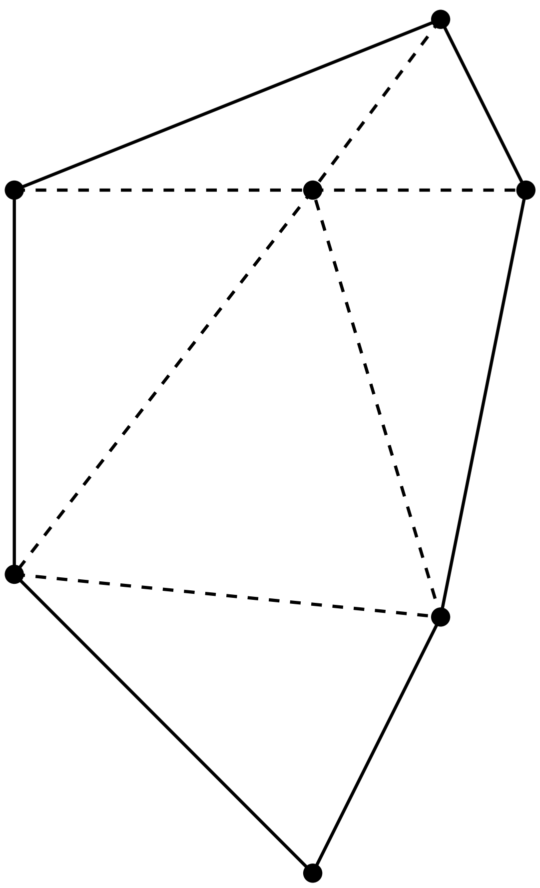

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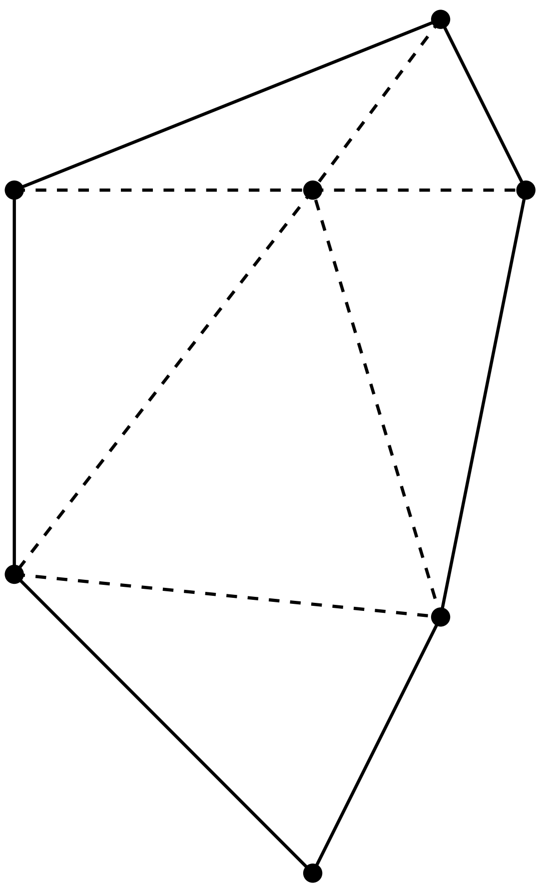

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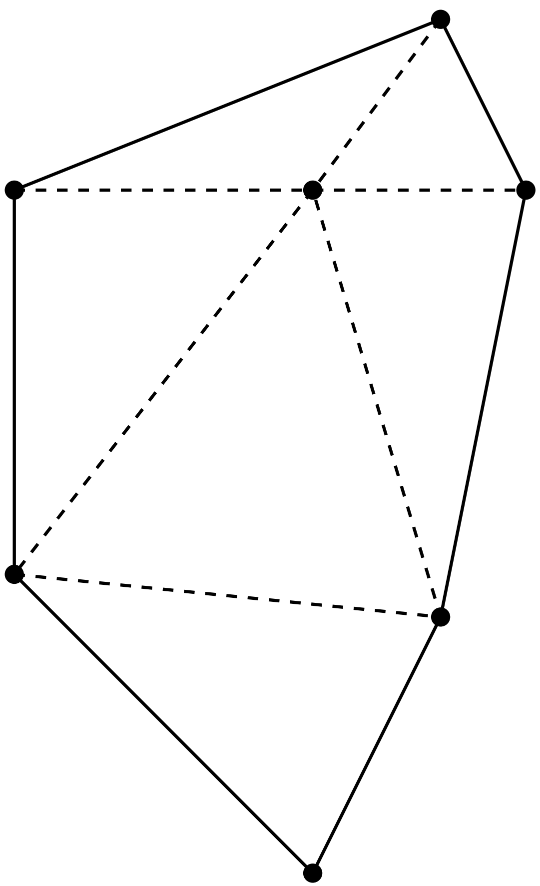

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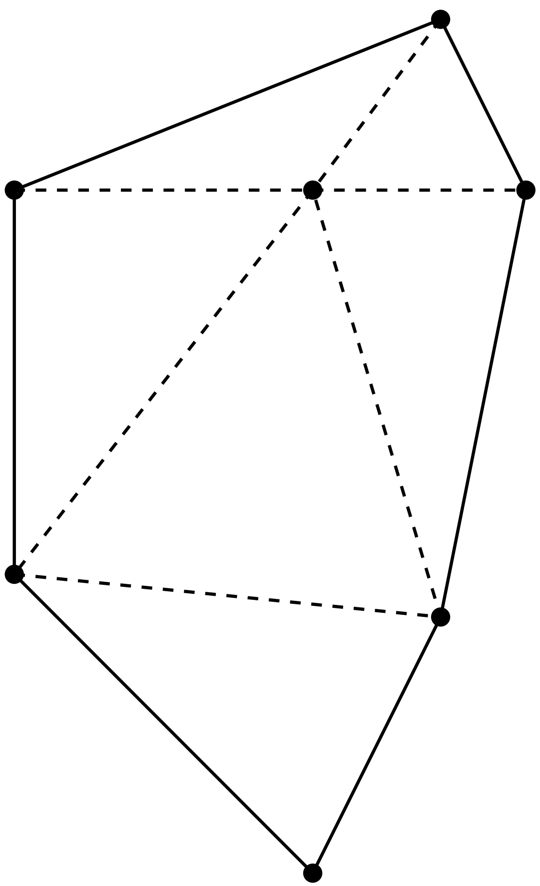

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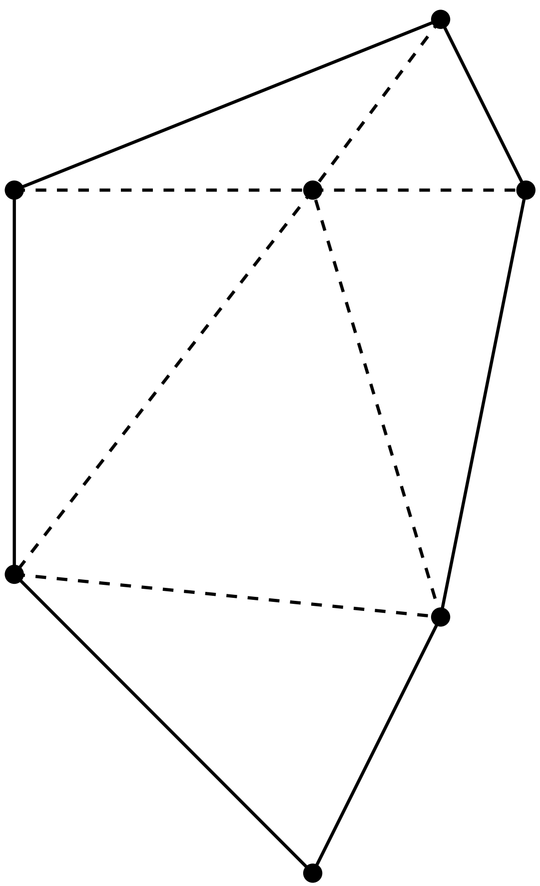

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Simplicial Cones

$S = \mathrm{cone}(x_1, \ldots, x_d)$ simplex. Then

$$E = \underbrace{\{q_1 x_1 + \cdots + q_d x_d \mid 0 \leq q_i < 1\}}_{\pi} \cap \mathbb{Z}^d$$

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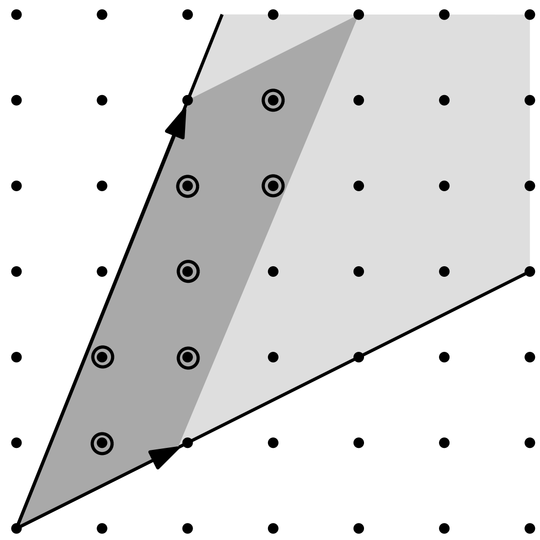

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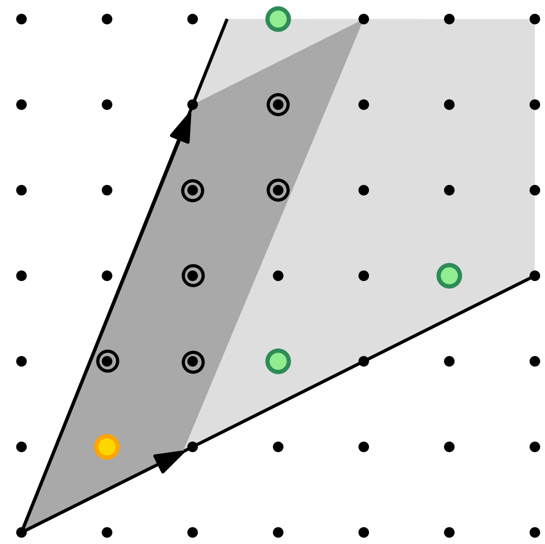

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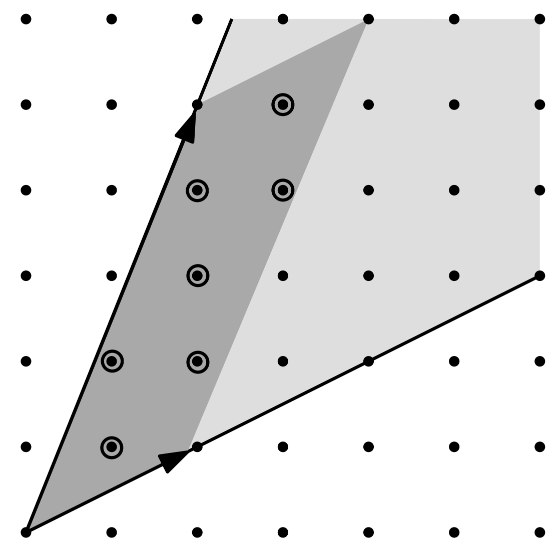

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Normaliz generates the points in E. They are candidates for the Hilbert Basis and their number is given by the volume of the simplex

$$|E| = \operatorname{vol}(S) = \det(x_1, \dots, x_d).$$

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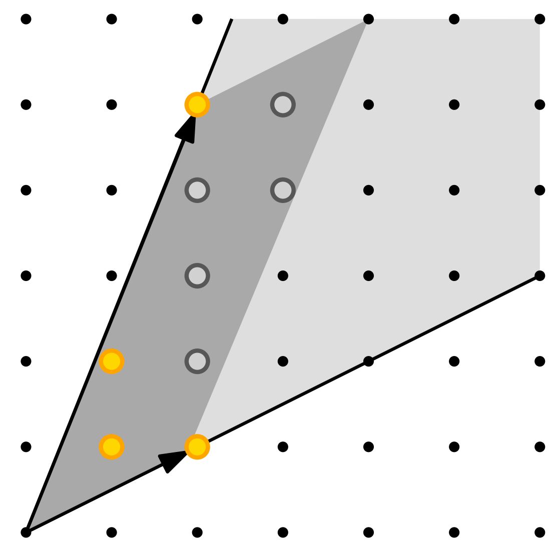

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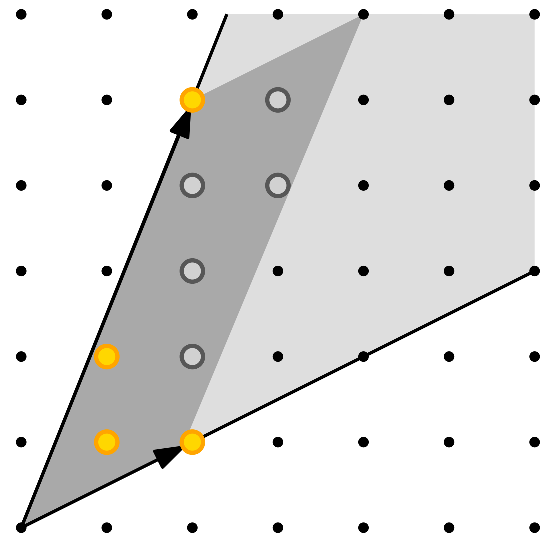

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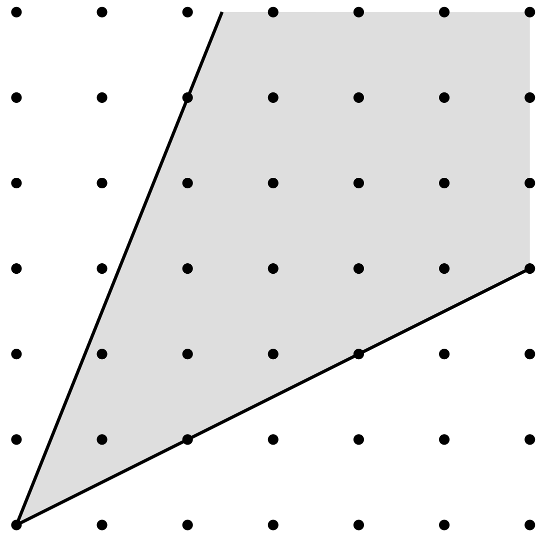
Therefore $\mathrm{vol}(S)$ is a critical size for the runtime of Normaliz.

Our Approach

If simplex S has big volume: decompose it into smaller simplices, such that the sum of their volumes decreases remarkably.



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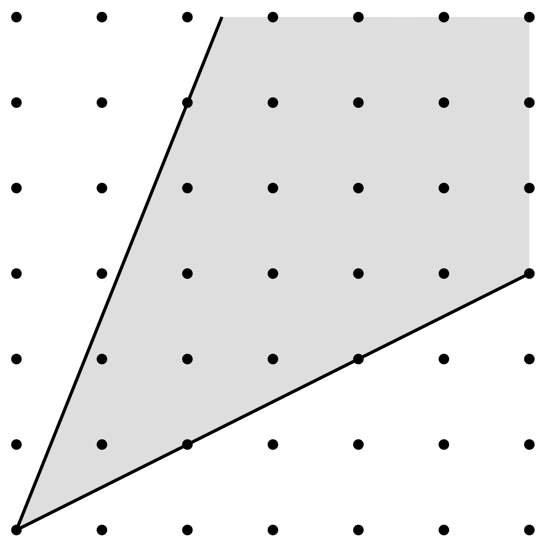

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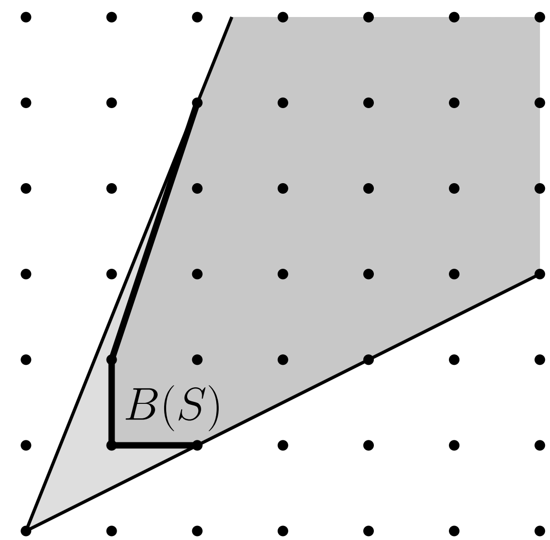

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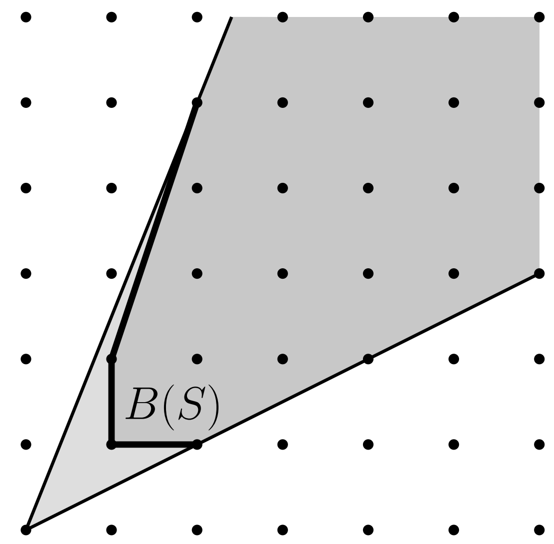

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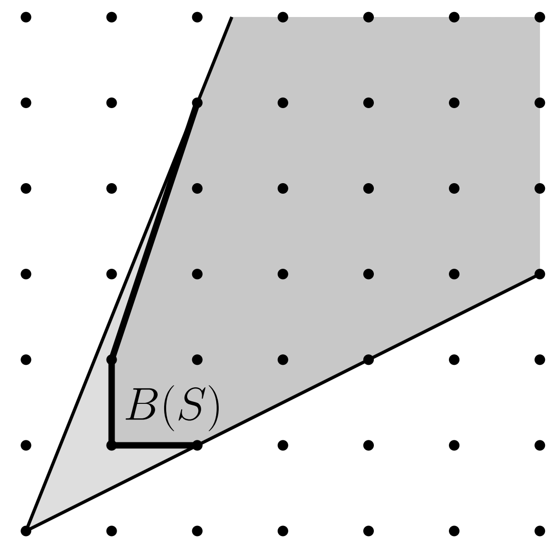

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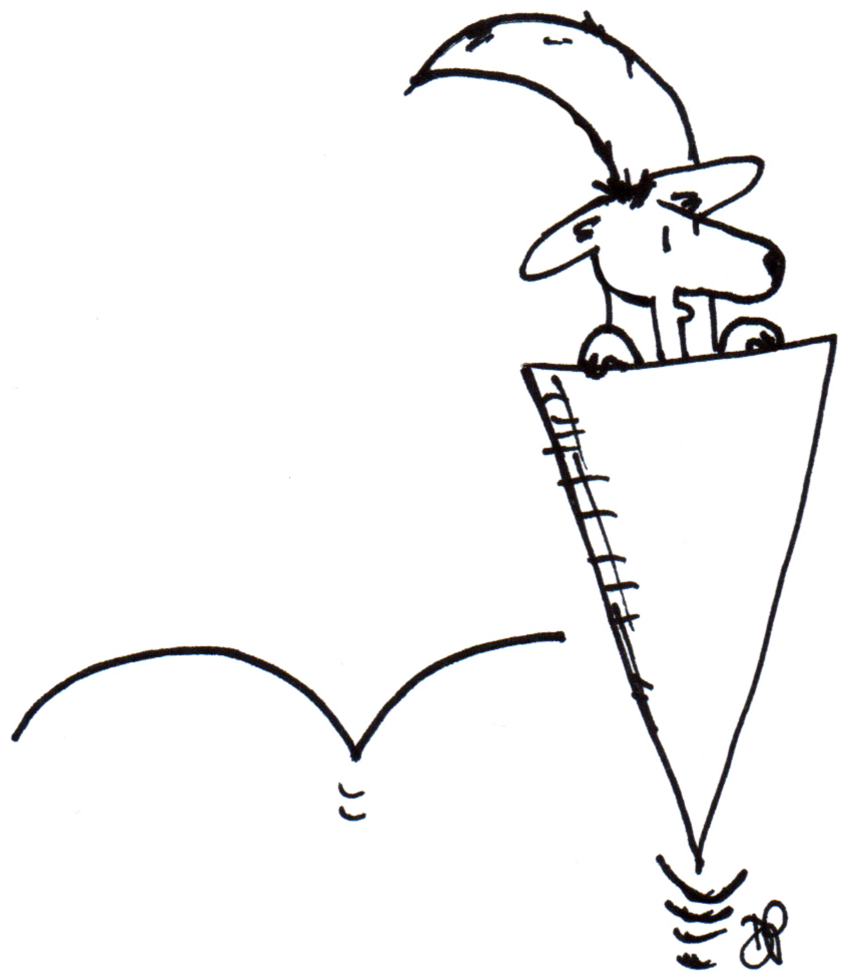
Determine only some points from $B(S)$ using

1. Integer Programming

2. Approximation

Integer Programming



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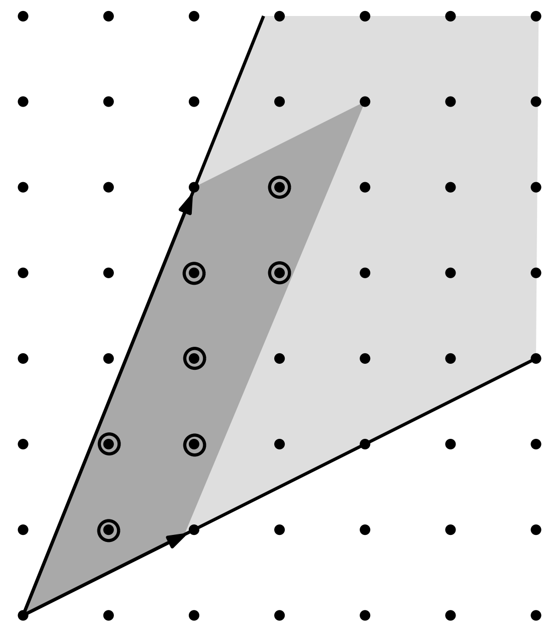
$S = \mathrm{cone}(x_1, \ldots, x_d)$ simplex in triangulation



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Compute a point x that minimizes the sum of determinants:

$$\sum_{i=1}^{d} \det(x_1, \ldots, x_{i-1}, x, x_{i+1}, \ldots, x_d) = N^T x,$$

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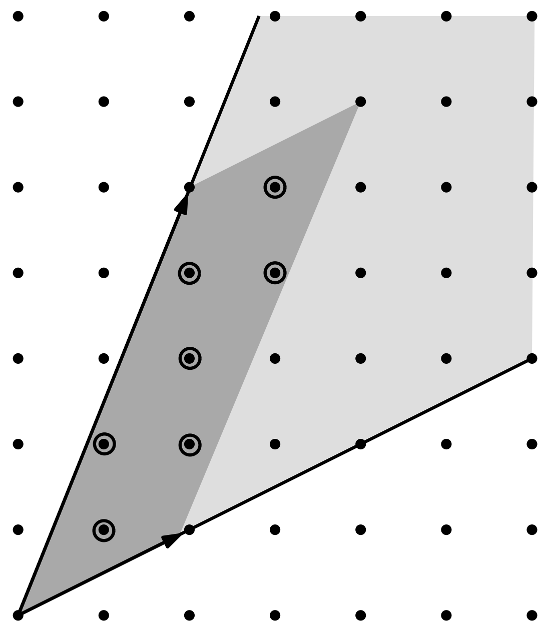

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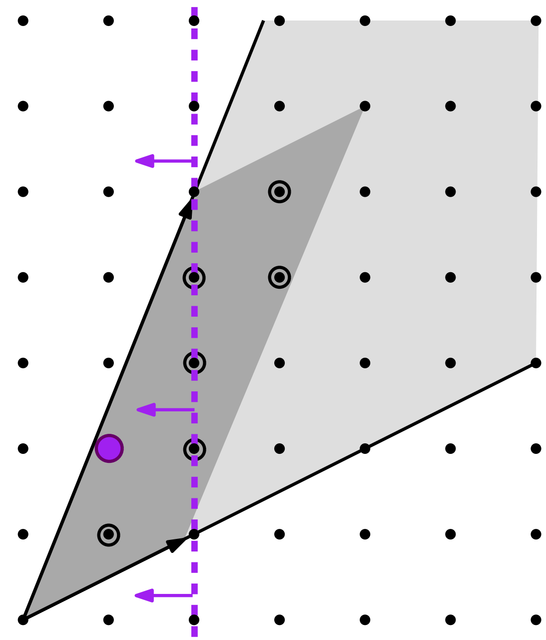

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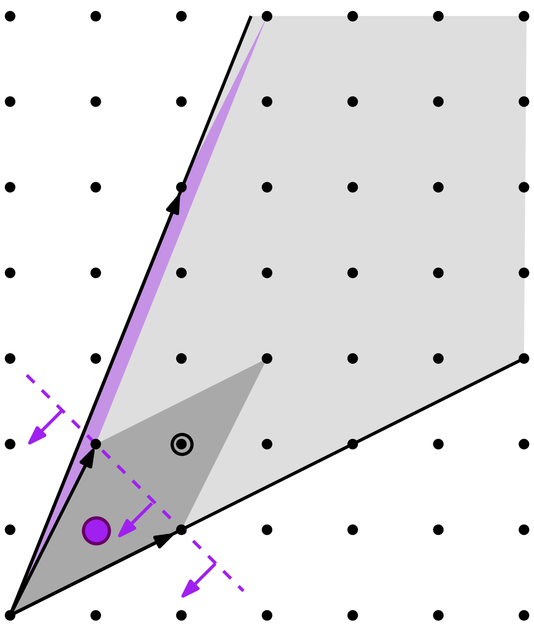

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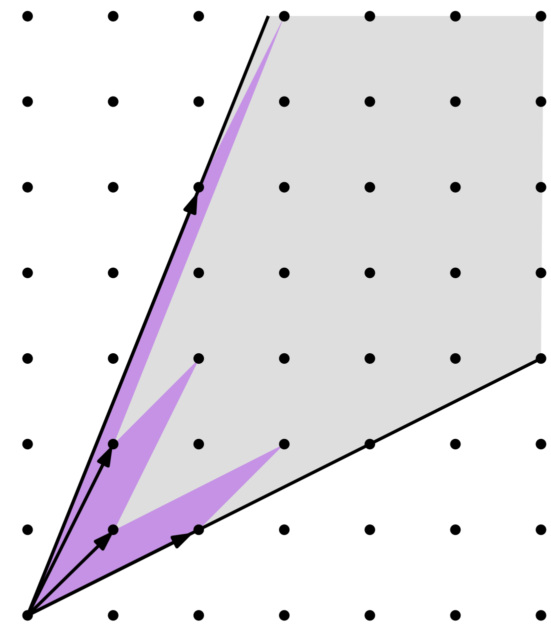

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Algorithm Bottom Points

Input: $S = \operatorname{cone}(x_1, \ldots, x_d)$ simplex with $\operatorname{vol}(S) \geq \text{Bound}$

Return: Points from $B(S)$

1: $\mathcal{B}, \mathcal{S} \leftarrow \emptyset$
2: store S into $\mathcal{S}$
3: **while** $\mathcal{S} \neq \emptyset$ **do**
4: let $T = \operatorname{cone}(y_1, \ldots, y_d)$ be the first element of $\mathcal{S}$ and delete it
5: compute a normal vector N on hyperplane spanned by $y_1, \ldots, y_d$
6: compute hyperplanes $\{H_1, \ldots, H_d\}$ and volume of T
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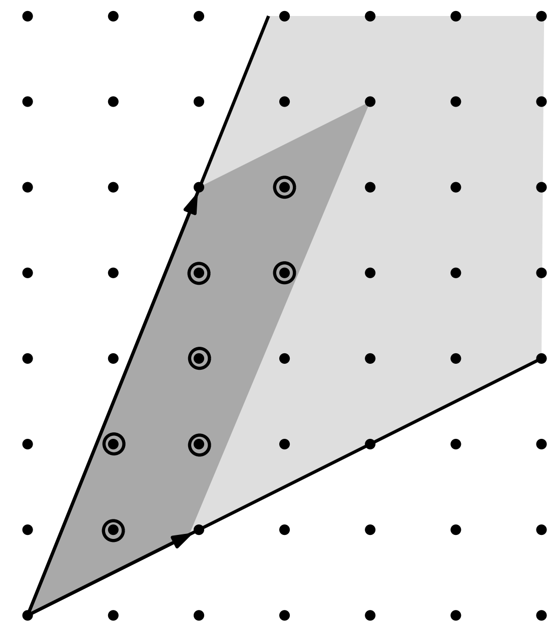

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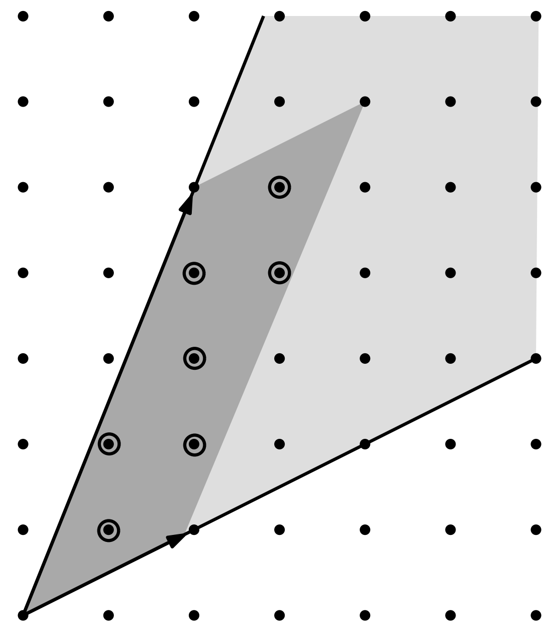

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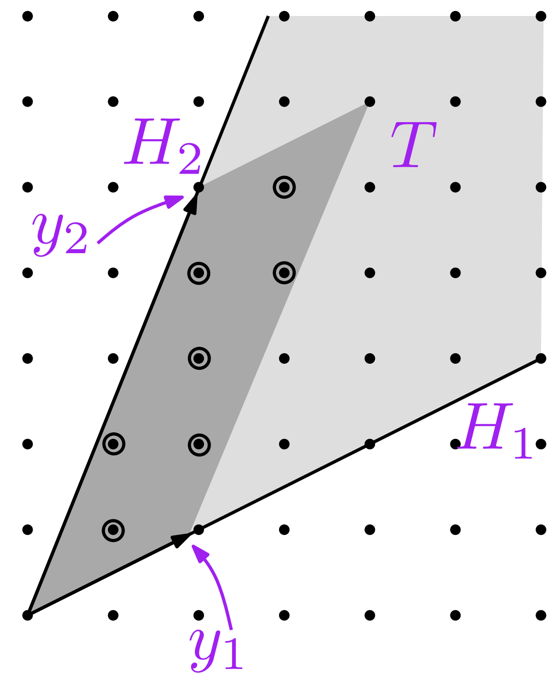
    if $\mathrm{vol}(T) < \mathrm{Bound}$ **then continue**



The Algorithm

Algorithm Bottom Points

while $\mathcal{S} \neq \emptyset$ **do**

$\vdots$

    if IP $(\star)$ is solvable for T **then**

        $y \leftarrow$ optimal solution of $(\star)$

        store y into $\mathcal{B}$

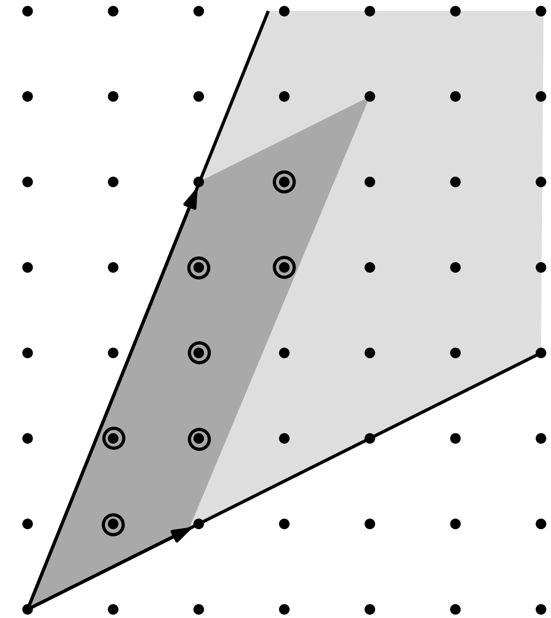
        for all hyperplanes H_i of T **do**

            if $y \notin H_i$ **then**

      $T_i \leftarrow \mathrm{cone}(y_1, \ldots, y_{i-1}, y, y_{i+1}, \ldots, y_d)$

                store T_i into $\mathcal{S}$

return $\mathcal{B}$



$$\boxed{\min\{N^T x \mid x \in S \cap \mathbb{Z}^d, x \neq 0, N^T x < N^T x_1\}} \qquad (\star)$$

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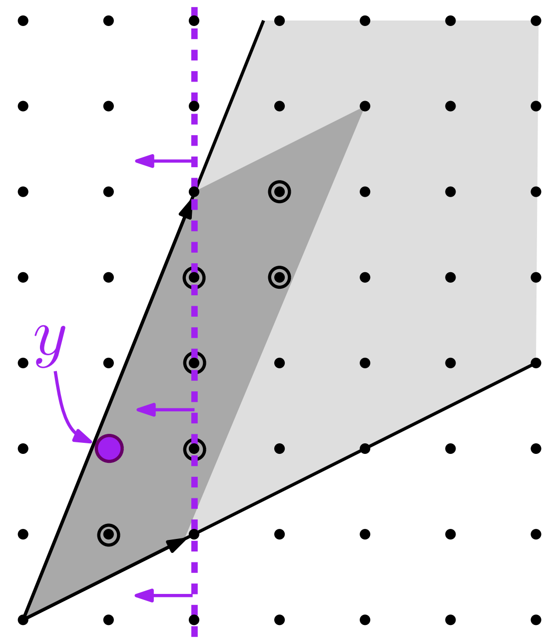

$\mathcal{B} = \{(1,2)$

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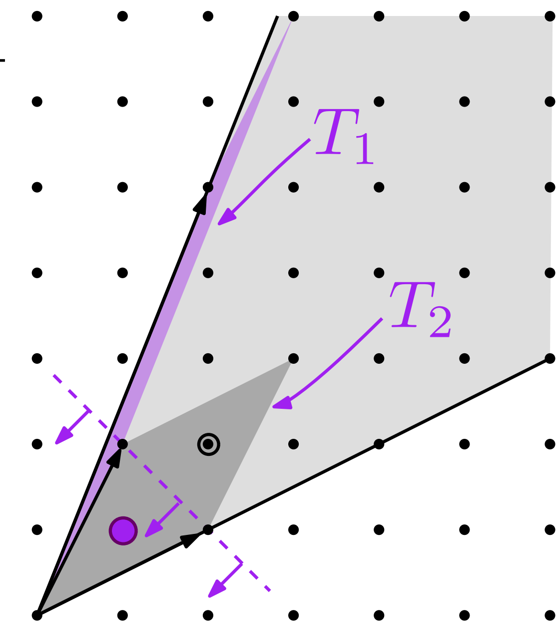

$$\mathcal{B} = \{(1,2), (1,1)\}$$

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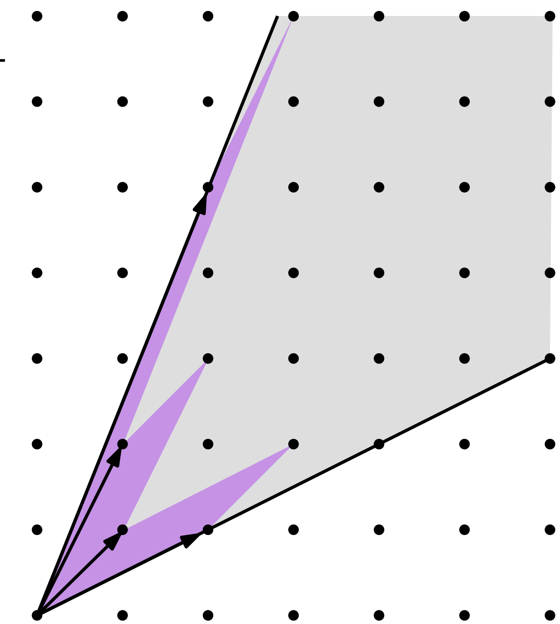

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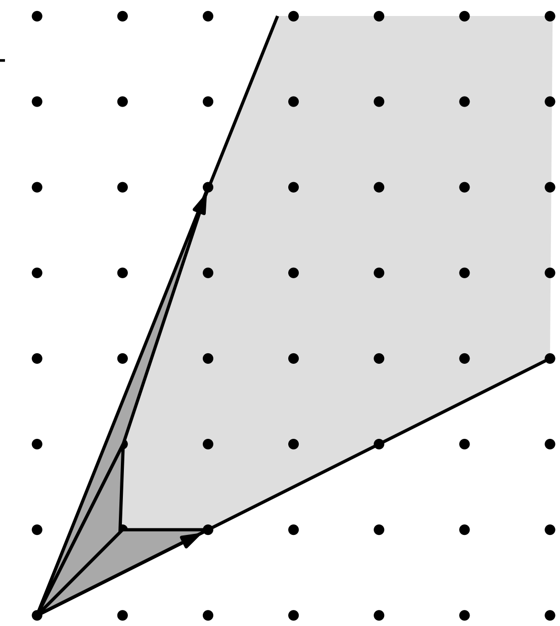

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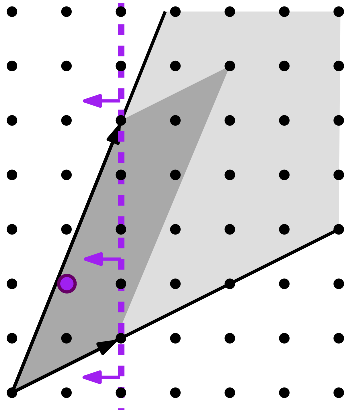
We triangulate the lower facets of $\mathrm{conv}(\mathcal{B} \cup \{x_1, \dots, x_d\})$ and evaluate this triangulation with the usual Normaliz algorithm.

The Algorithm

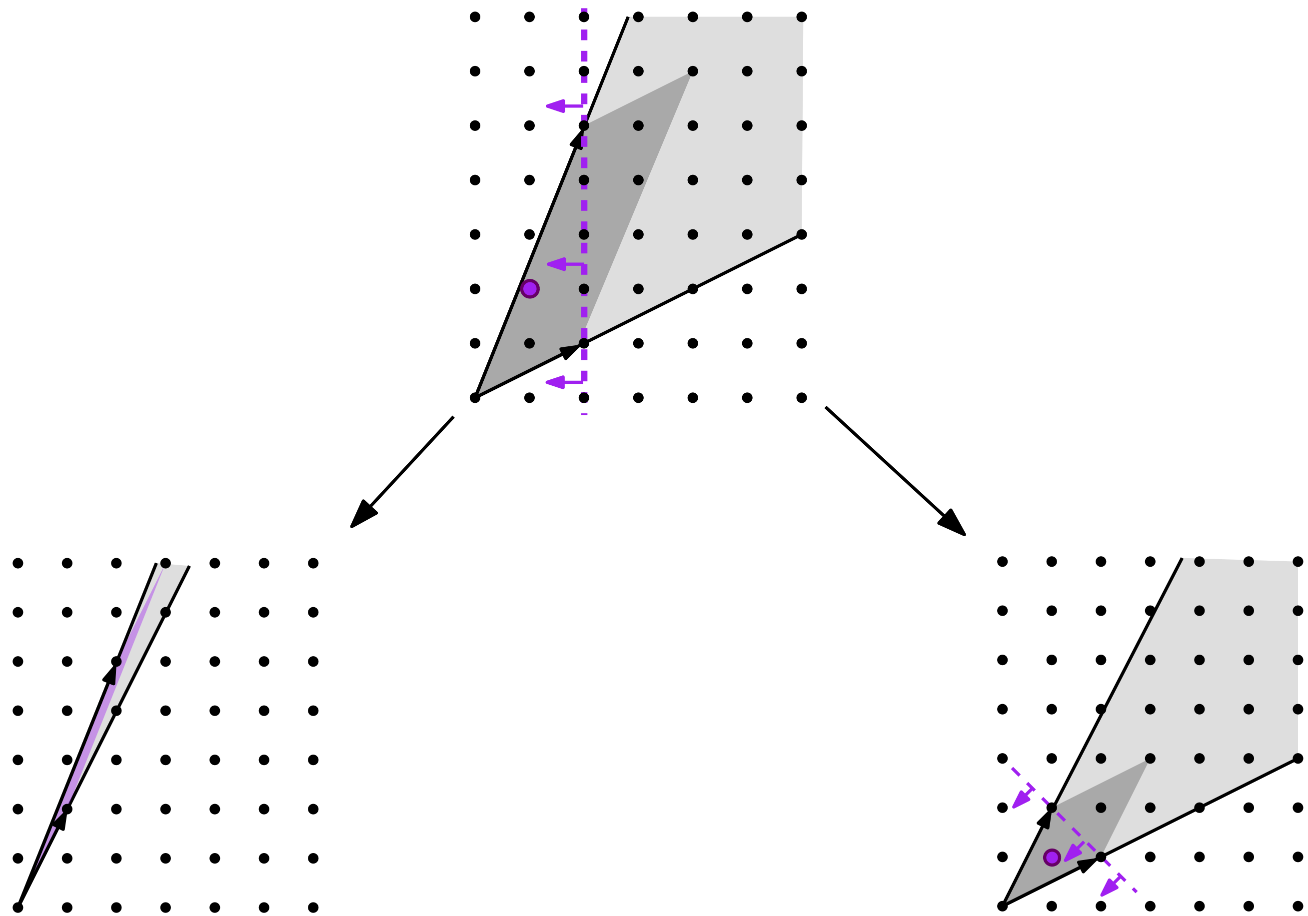
Level 0



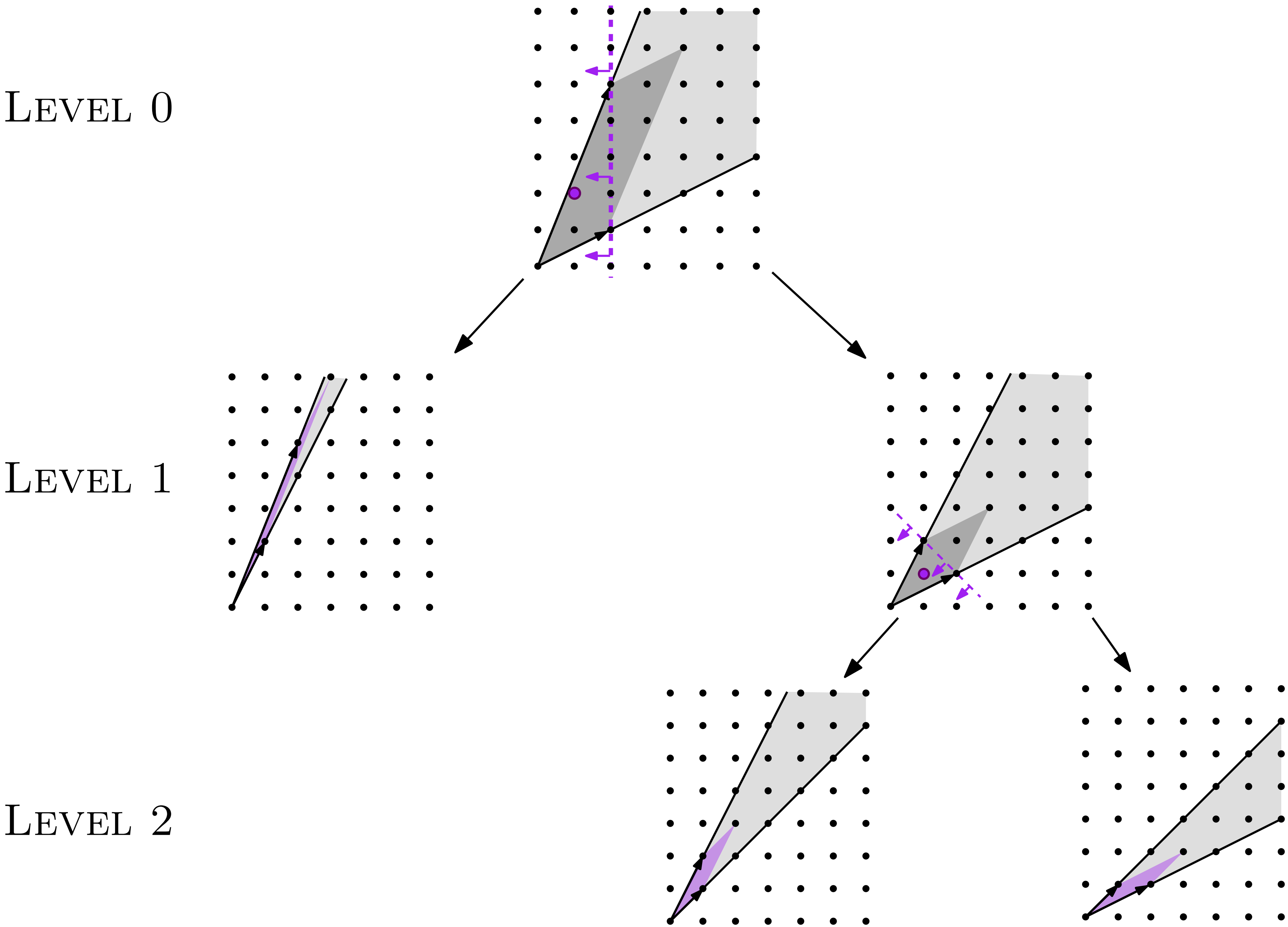
The Algorithm

Level 0

Level 1

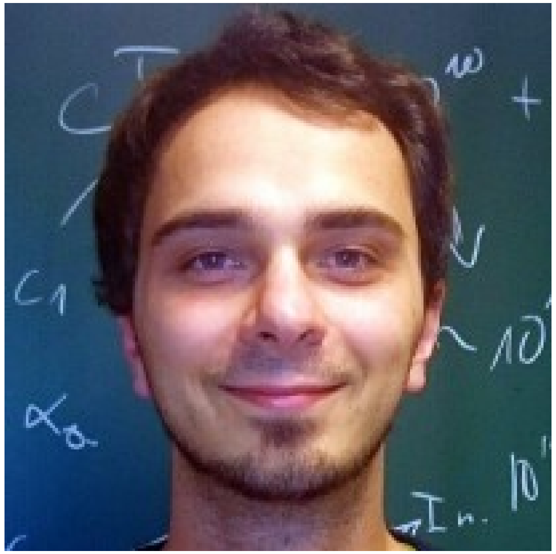


The Algorithm



Implementation & Results

- use SCIP (3.2.0) via its C++ interace



Gregor Hendel

Implementation & Results

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  - individual time limit
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volume used			
integer programs solved			
improvement factor			
old runtime			
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SUN xFire 4450, 4 Intel Xeon X7460 processors, 20 threads, $\mathrm{SCIPBound} = 10^6$

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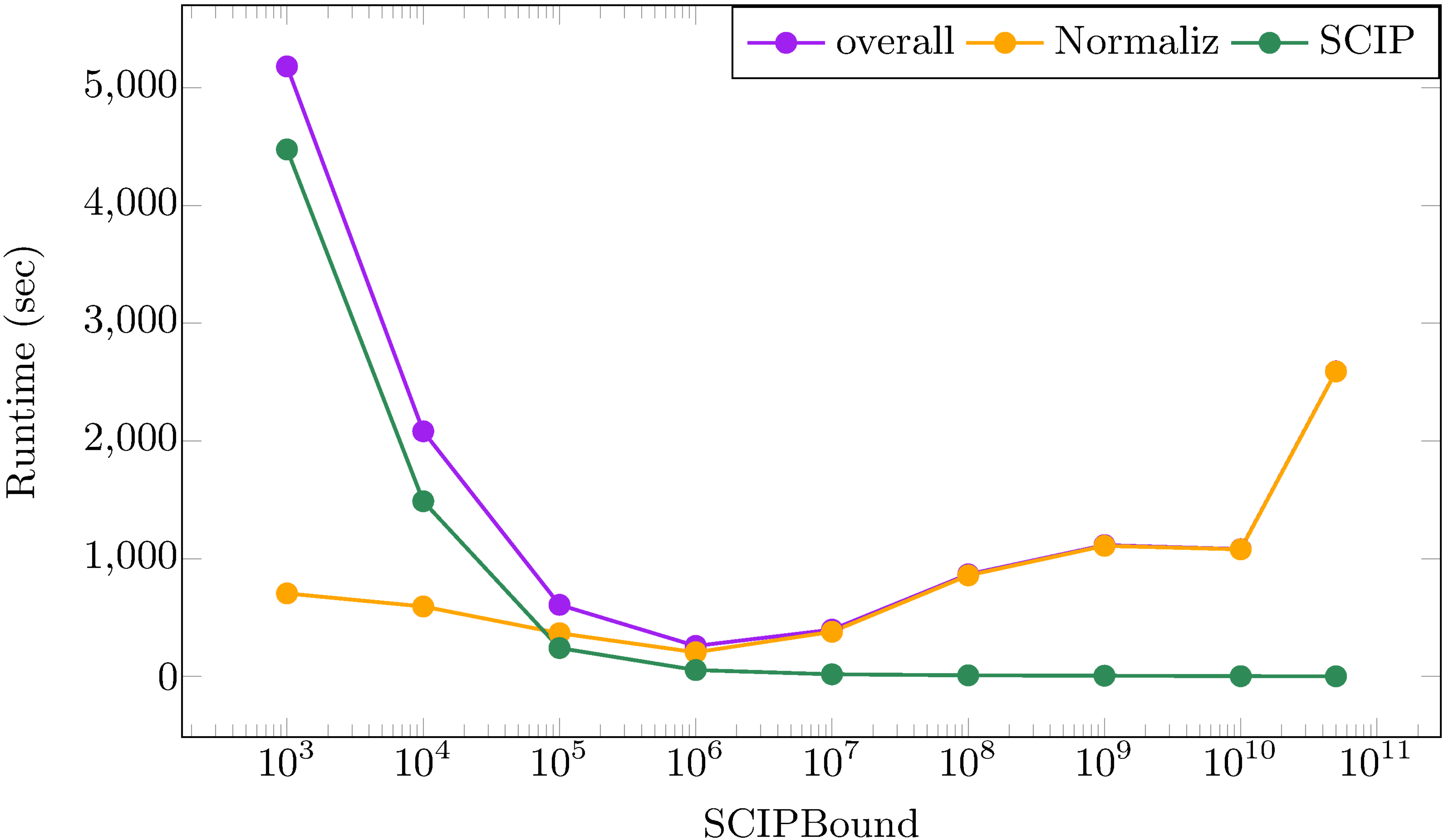
Implementation & Results



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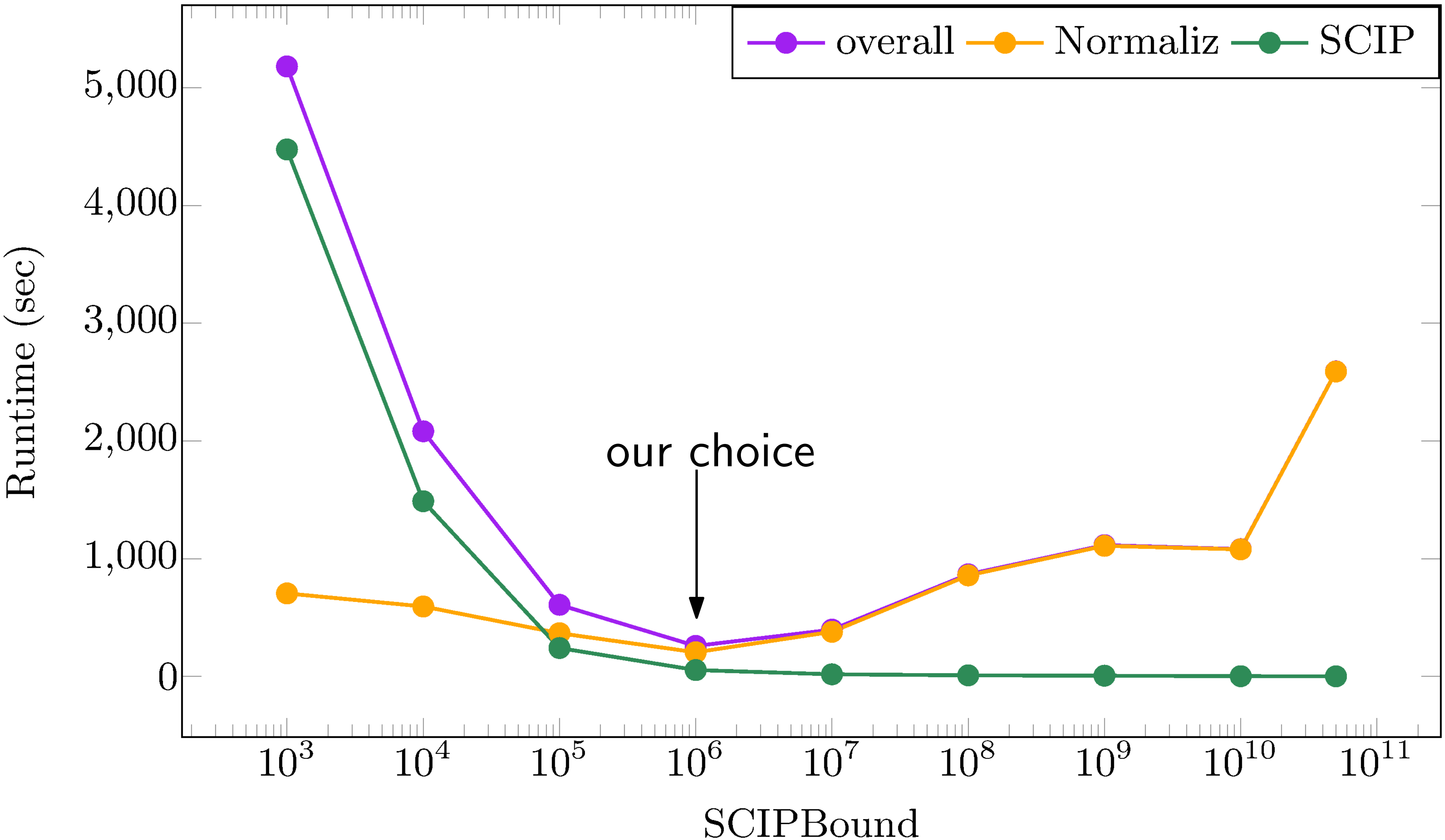
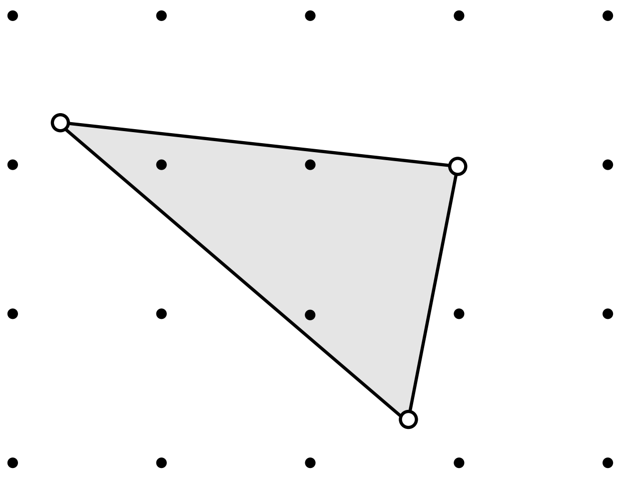


Approximation

The Algorithm

1. Look at the cross section at level 1 of the (transformed) simplex.



cross section at level 1

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1. Look at the cross section at level 1 of the (transformed) simplex.

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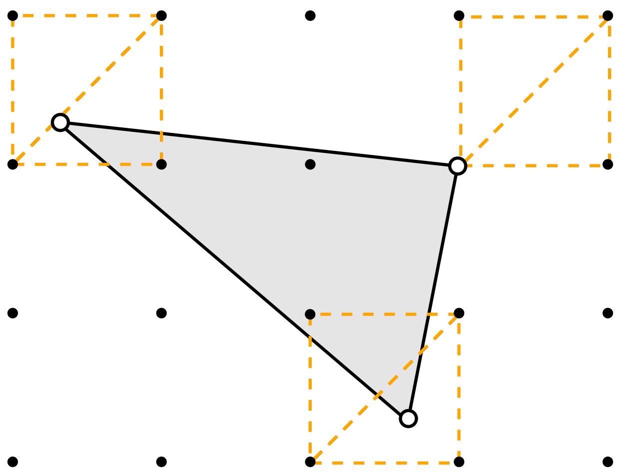

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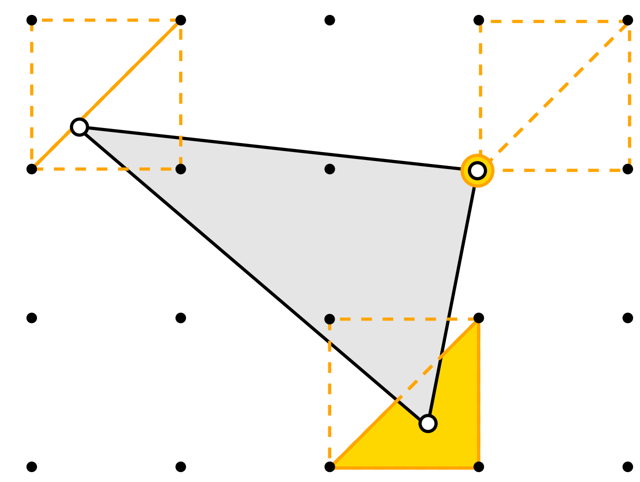

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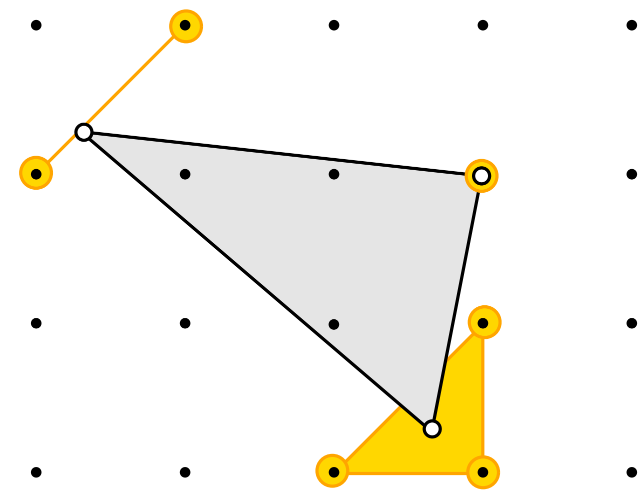

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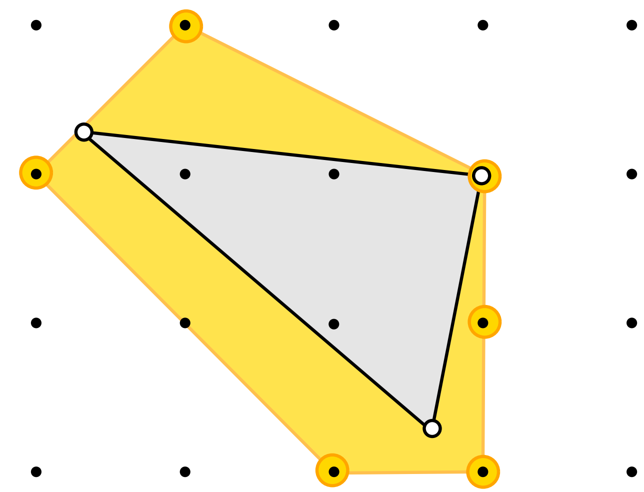

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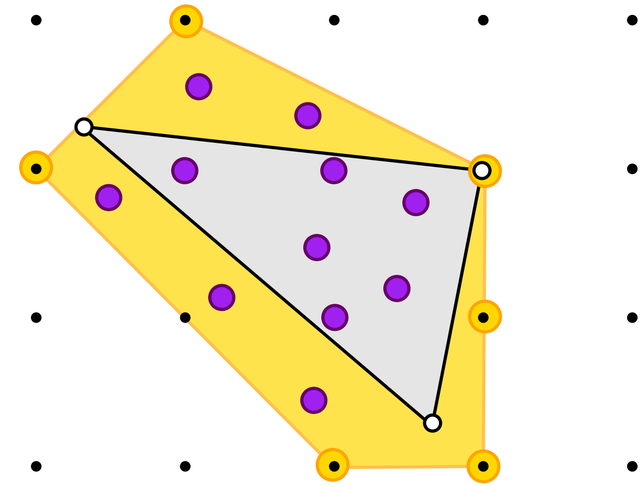

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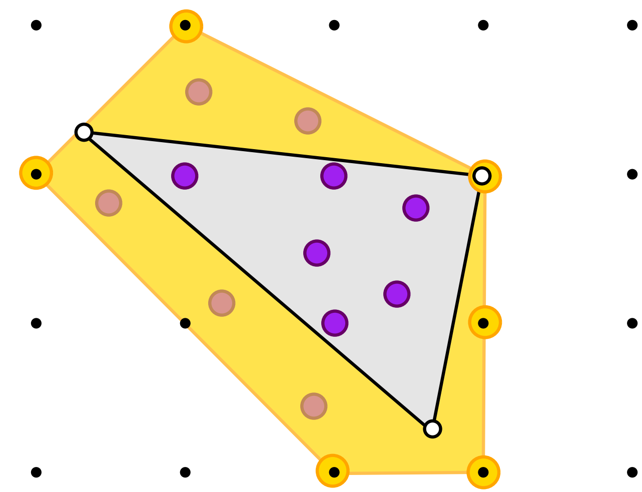

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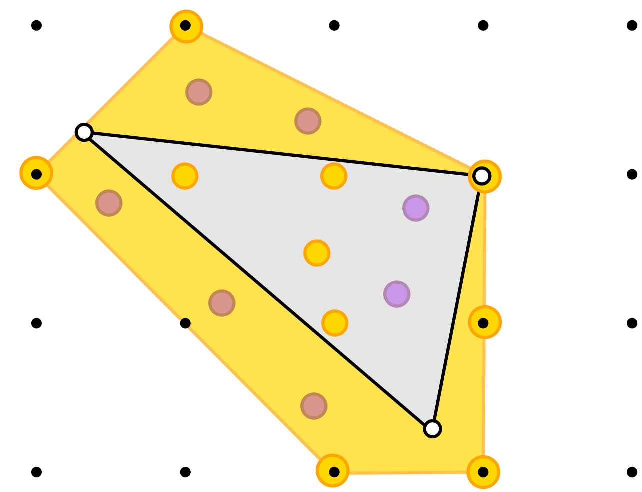

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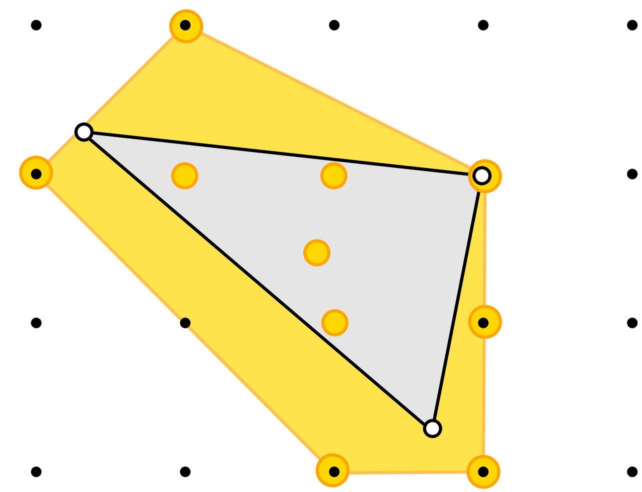
$\Rightarrow$ list of points $\mathcal{B}$ (bottom candidates)



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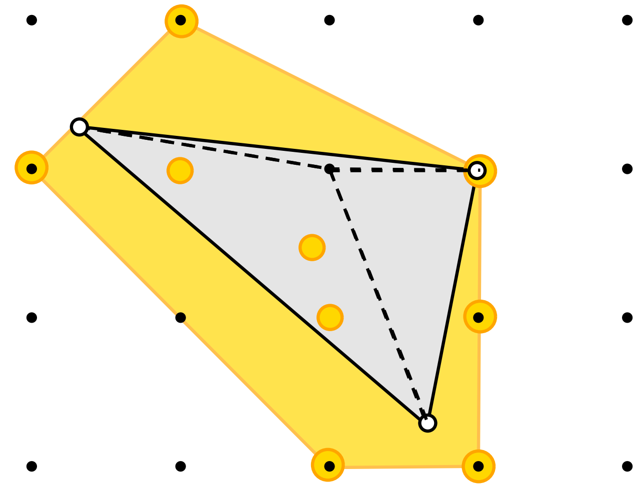

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Choose a grading minimizing point from $\mathcal{B}$ and continue as before.

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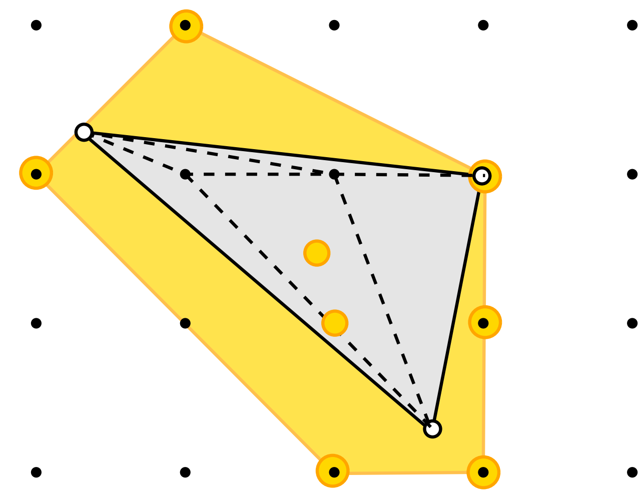

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Results

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	(1)	(2)	(1)	(2)	(1)	(2)
our vol	$3.93\,\mathrm{e}\,6$	$3.93\,\mathrm{e}\,6$	$5.47\,\mathrm{e}\,7$	$8.42\,\mathrm{e}\,7$	$2.39\,\mathrm{e}\,7$	$9.36\,\mathrm{e}\,9$
factor	25	25	$7.62\,\mathrm{e}\,6$	$4.95\,\mathrm{e}\,6$	$1.09\,\mathrm{e}\,7$	$2.99\,\mathrm{e}\,4$
old time	2s		$>$12d		$>$8d	
new time	0.5s	0.4s	46s	50s	5s	2m30s

Improvements & Outlook

⋆ If "large" simplices are remaining (both cases): approximate on a higher level.
(Which one?)

Improvements & Outlook

- If "large" simplices are remaining (both cases): approximate on a higher level.
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- Tweak settings in SCIP (time bounds etc.).

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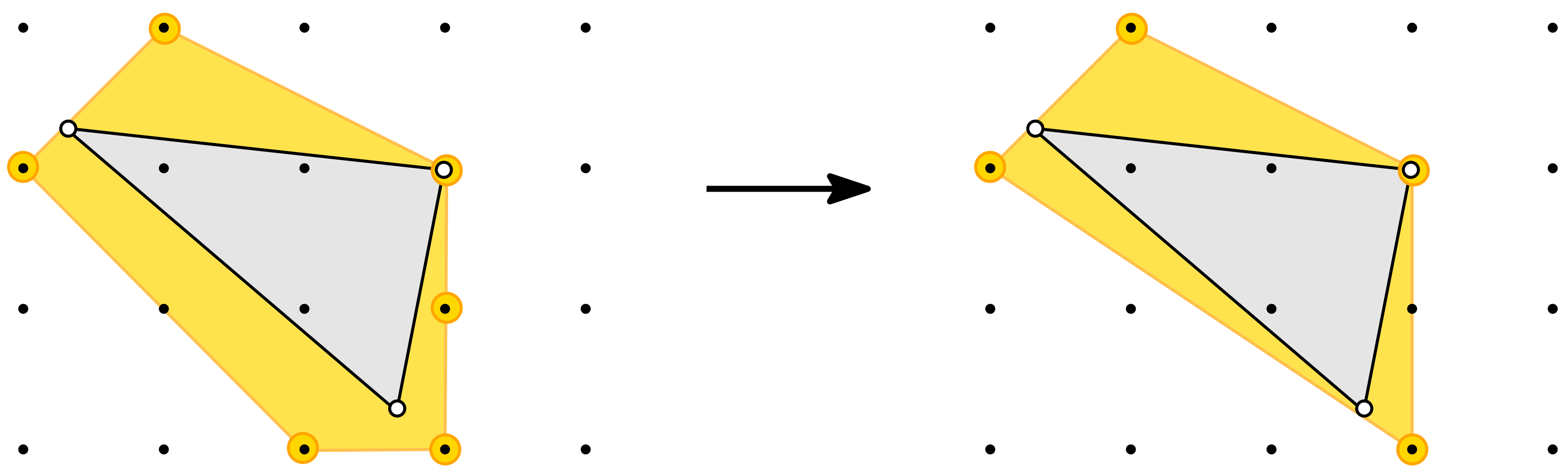
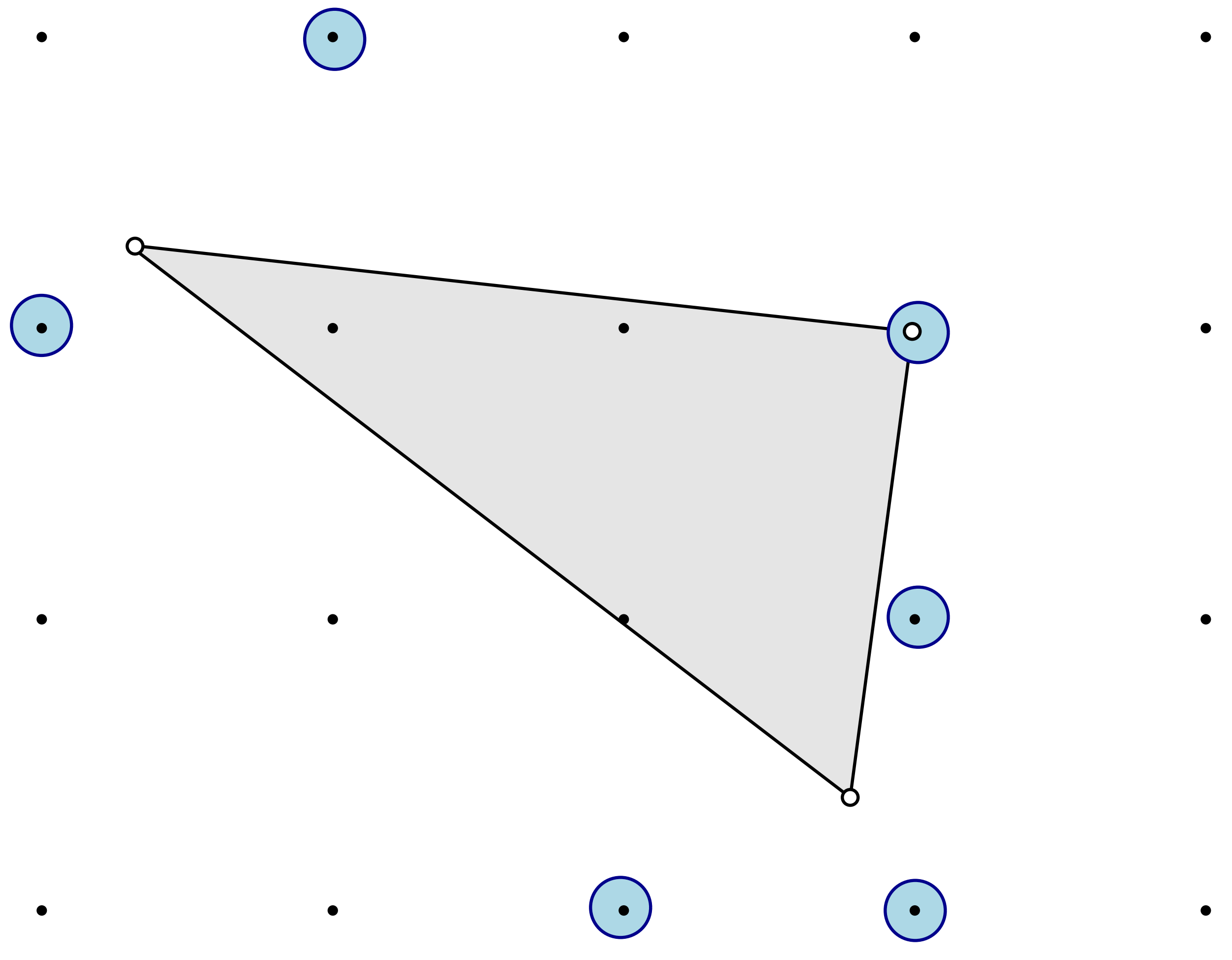
- Use less generators for approximating cone.
  (Partial Fourier-Motzkin Elimination)



Partial Fourier-Motzkin Elimination



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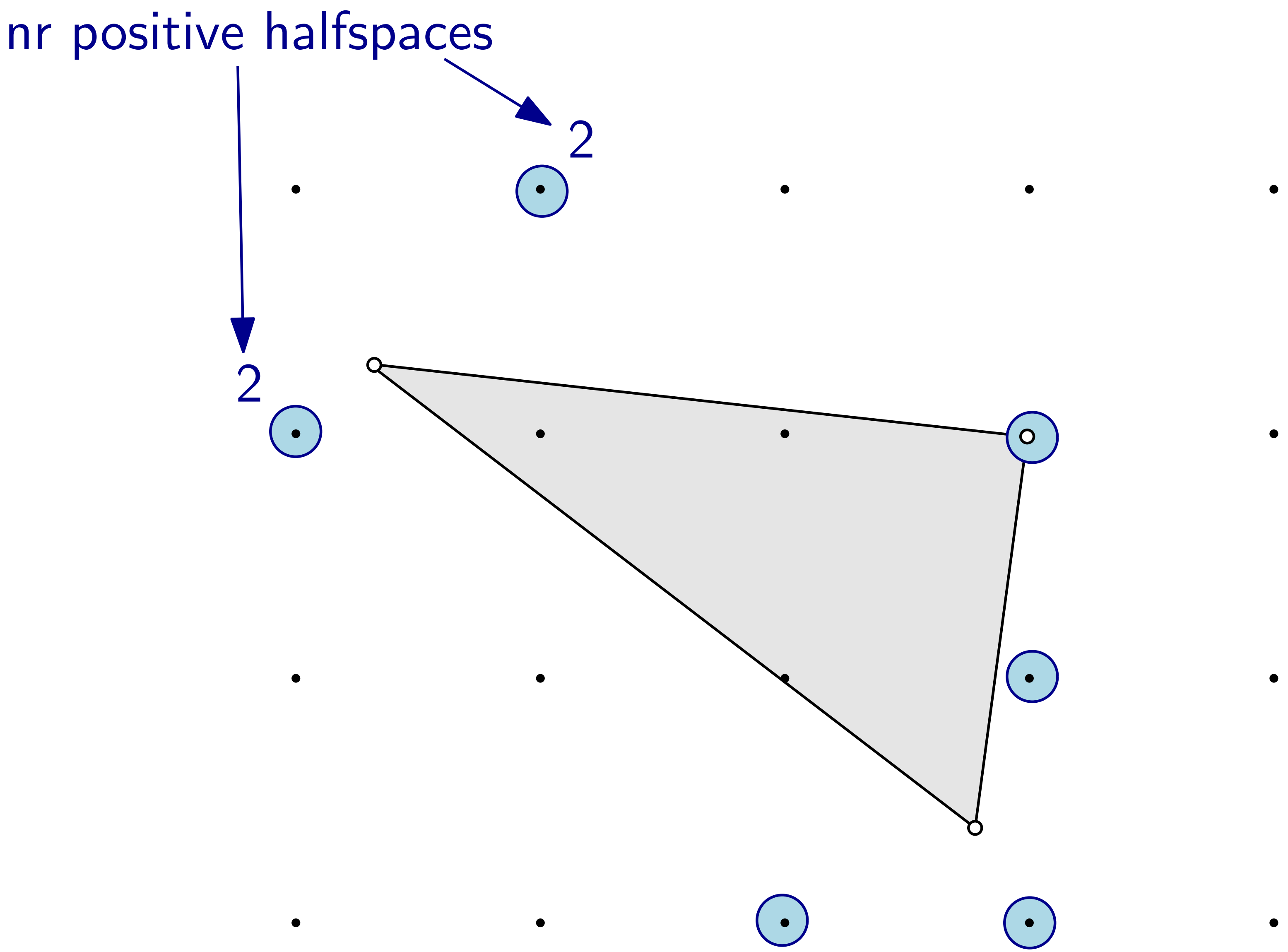

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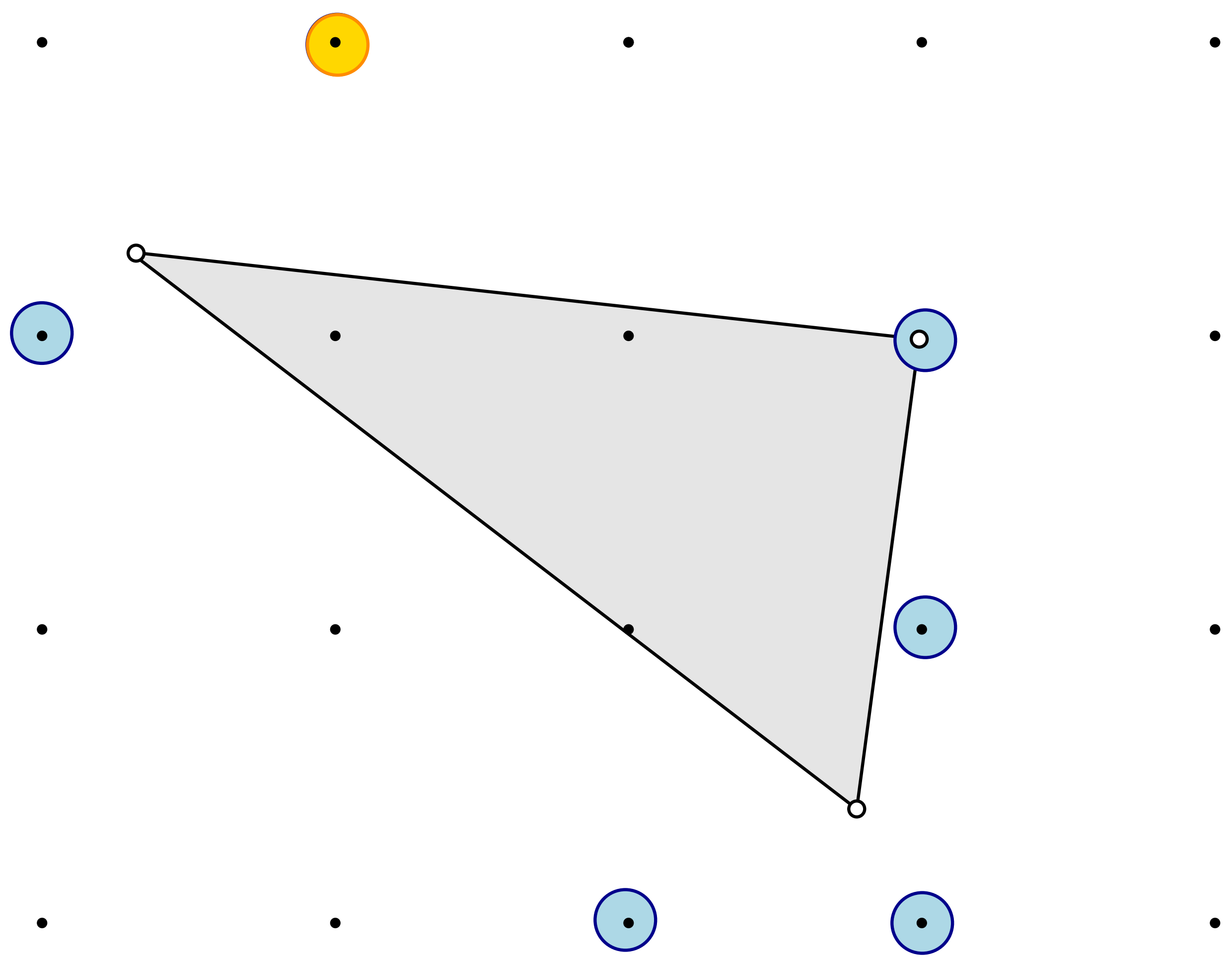

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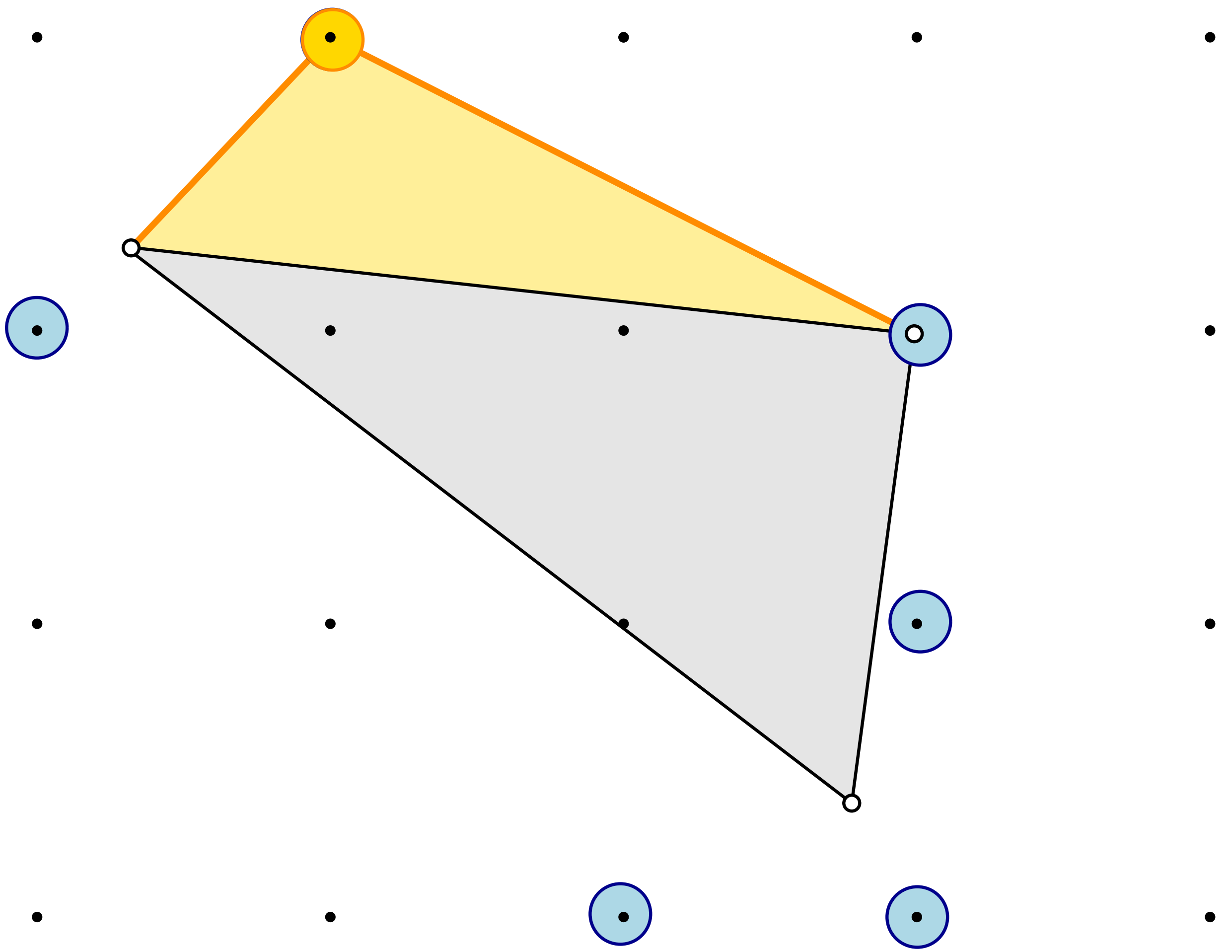

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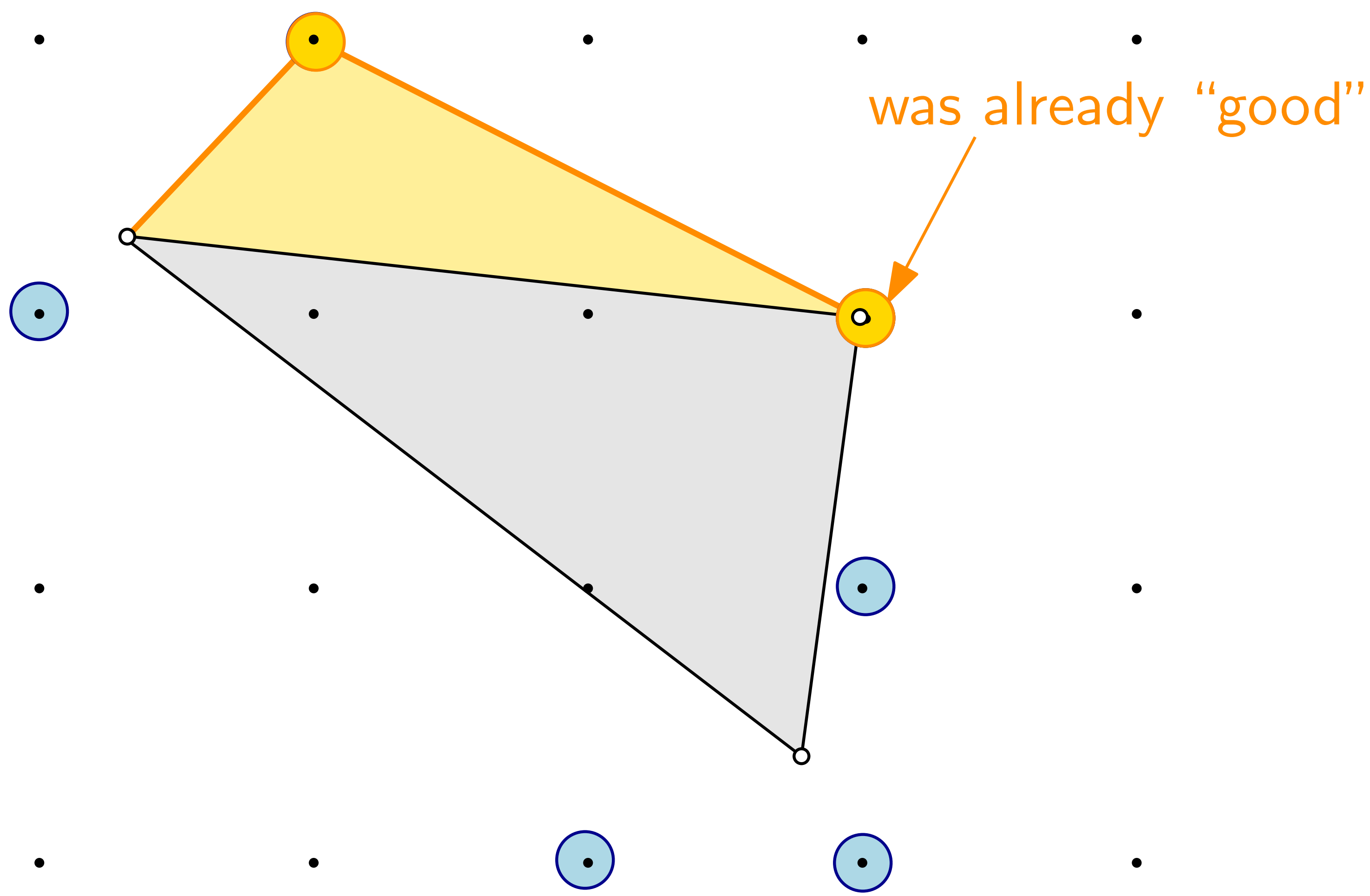

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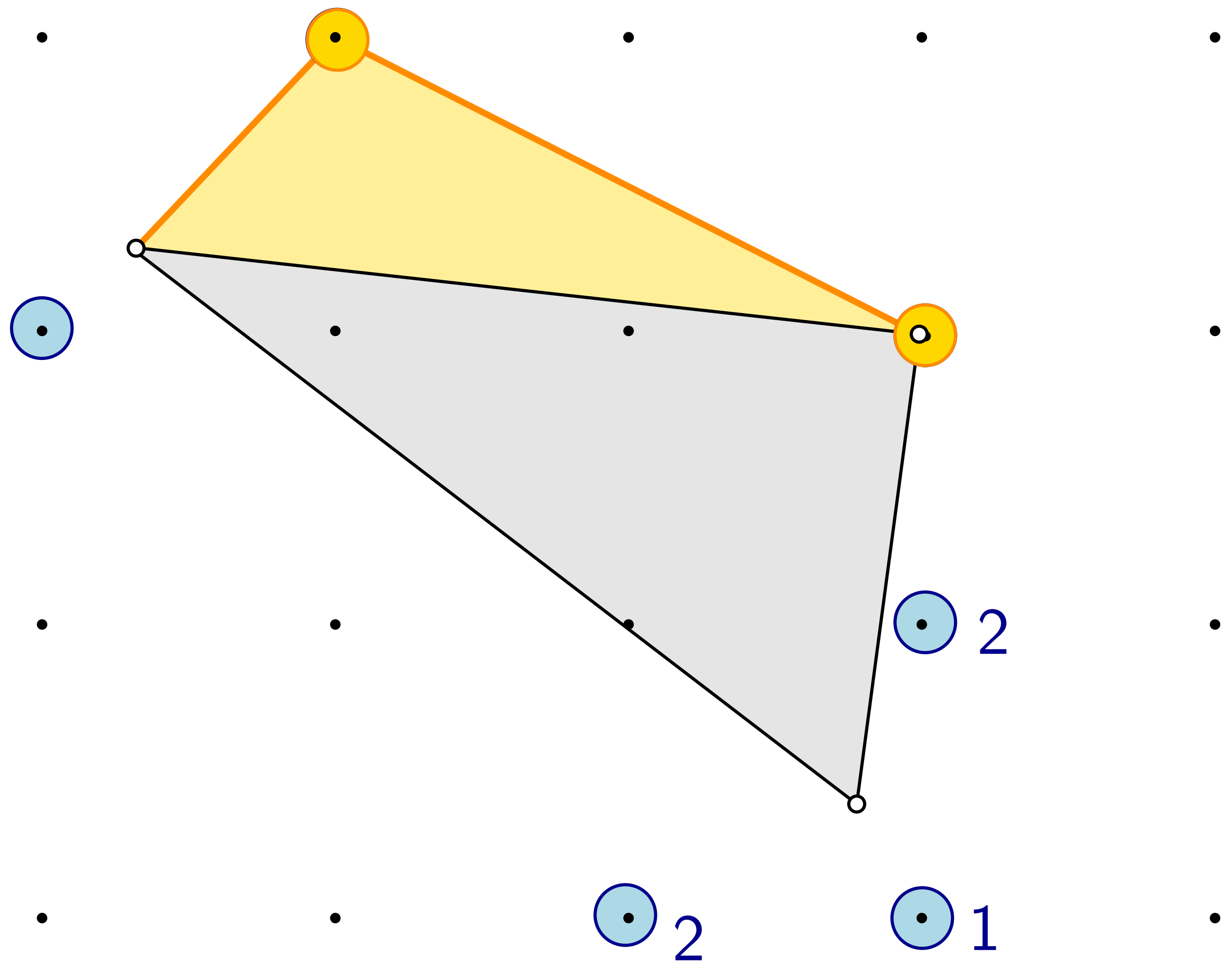

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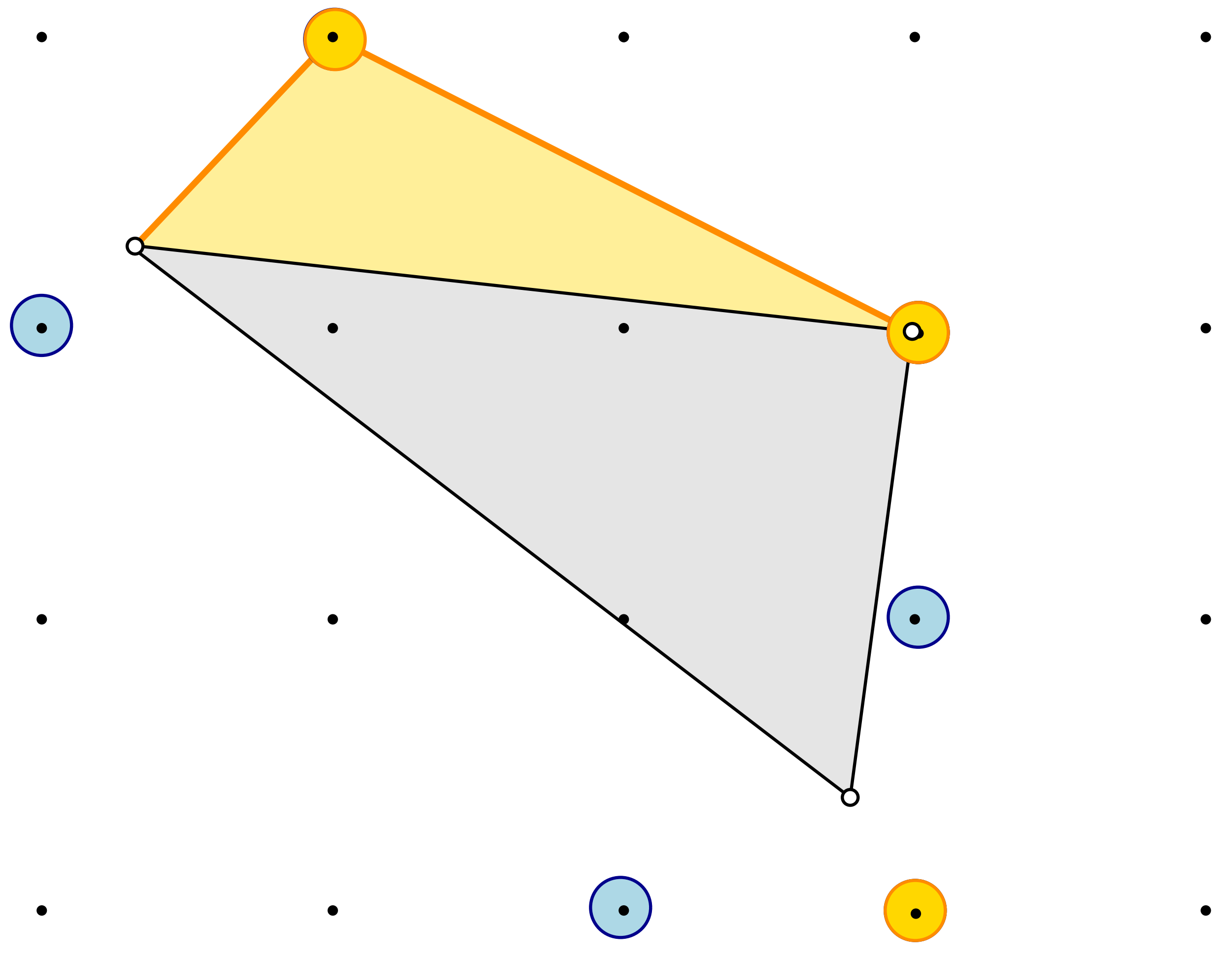

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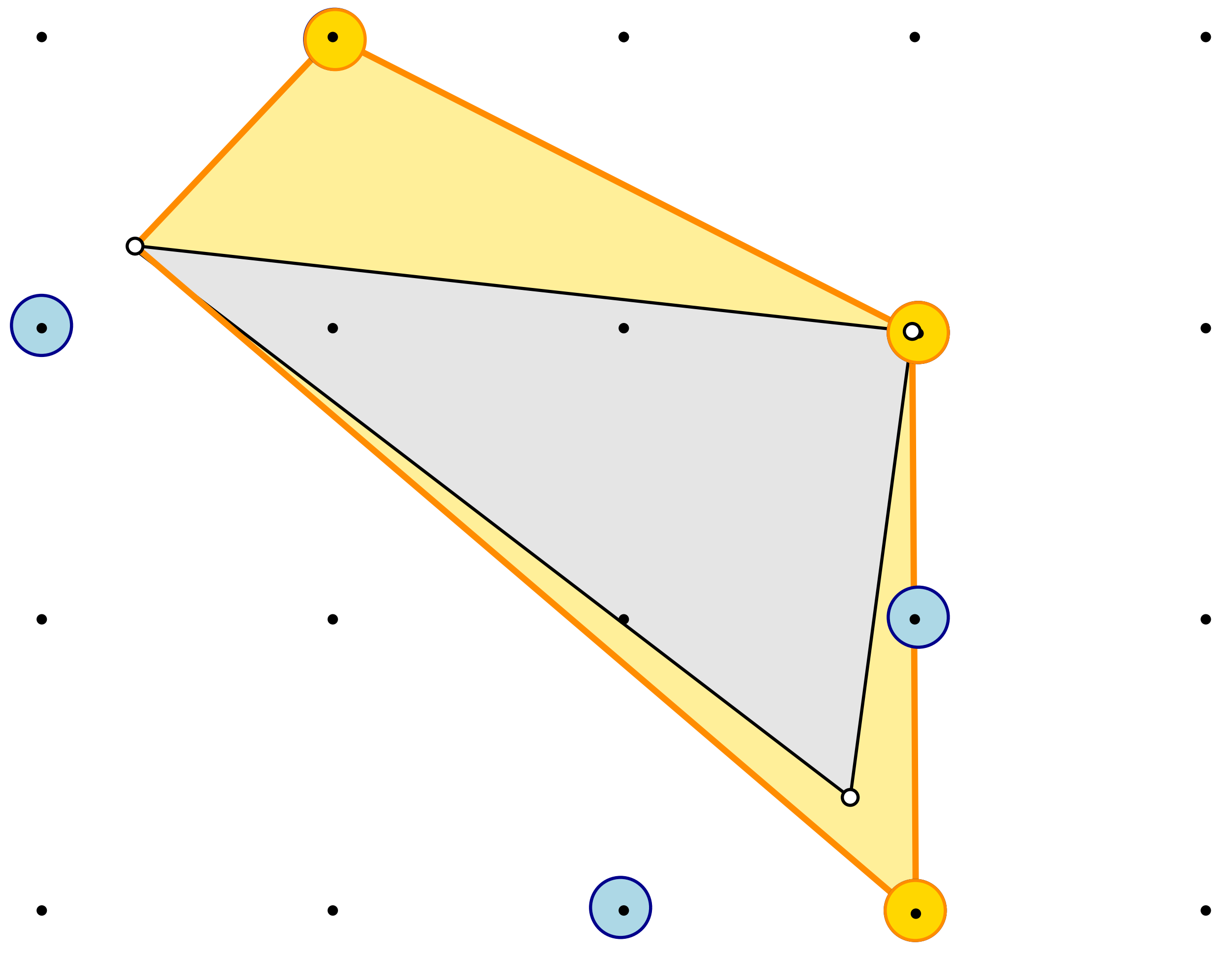

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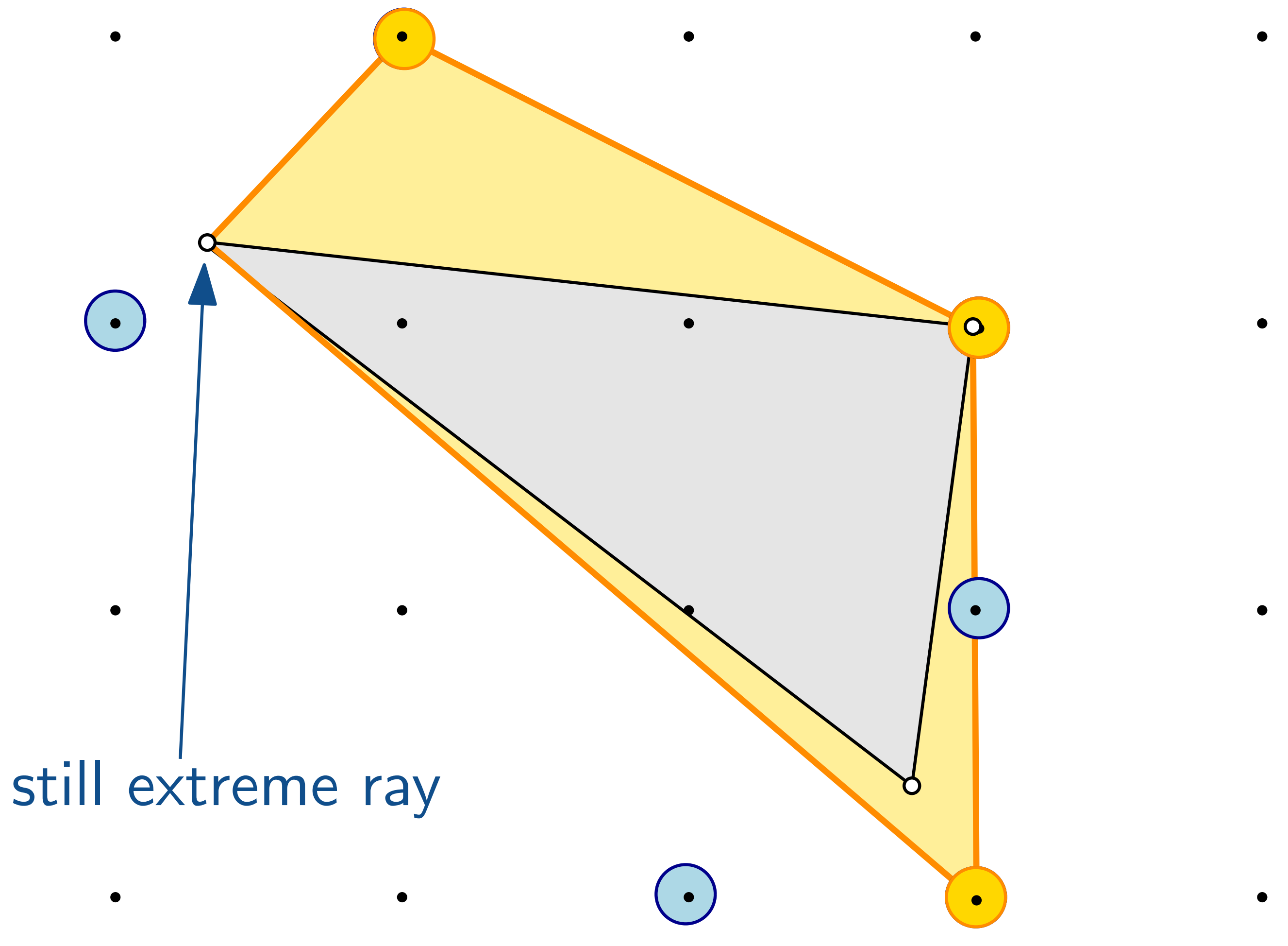

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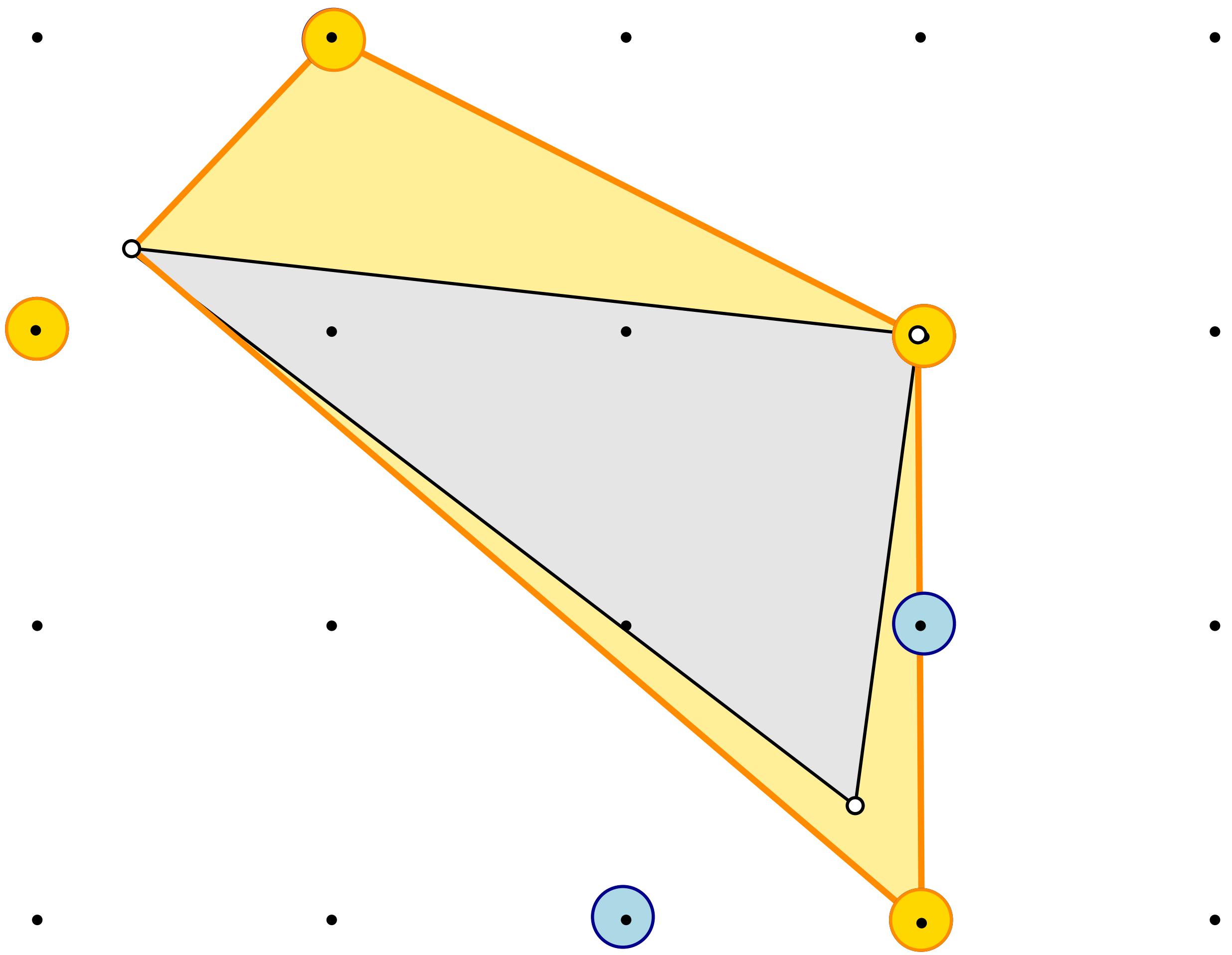

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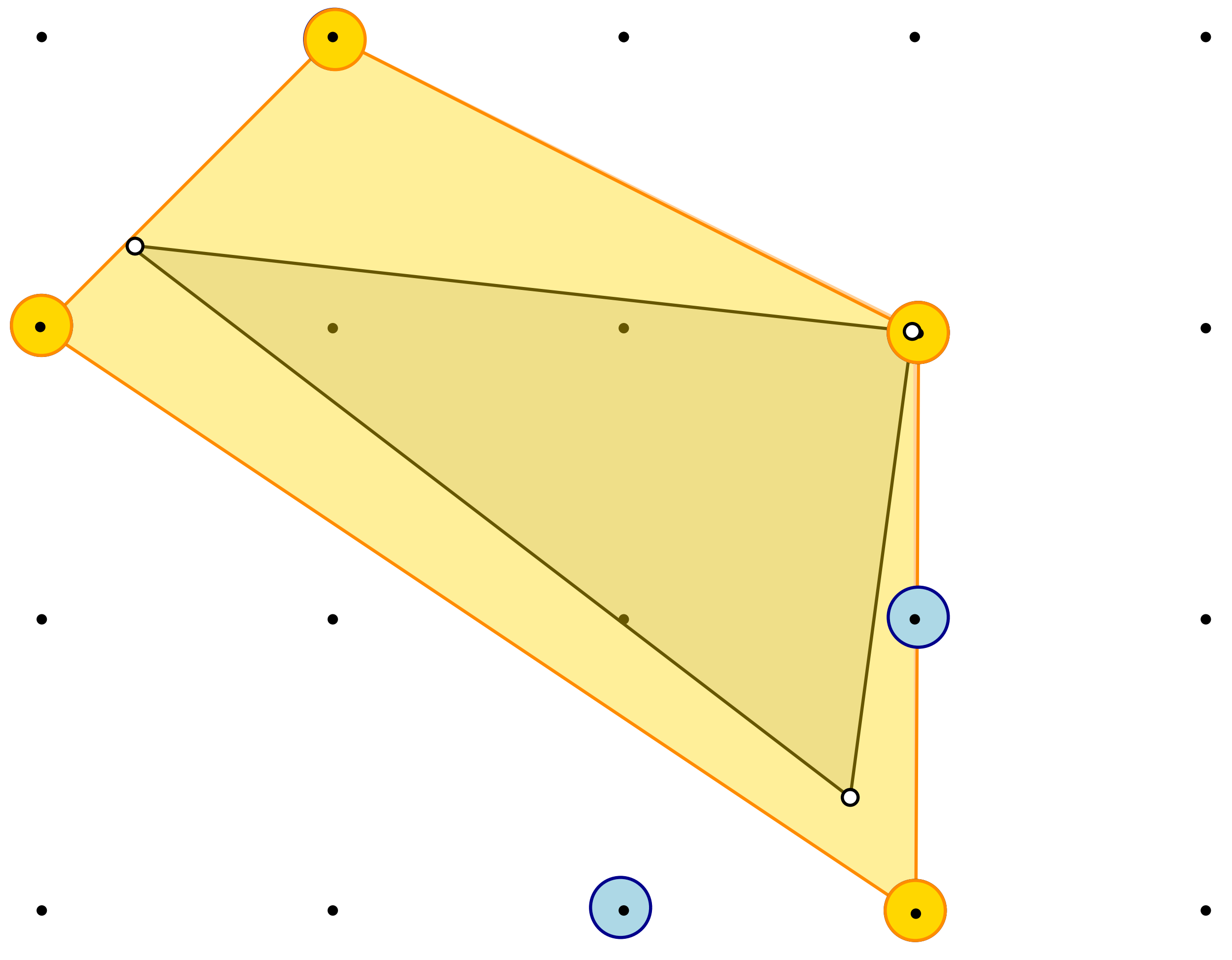
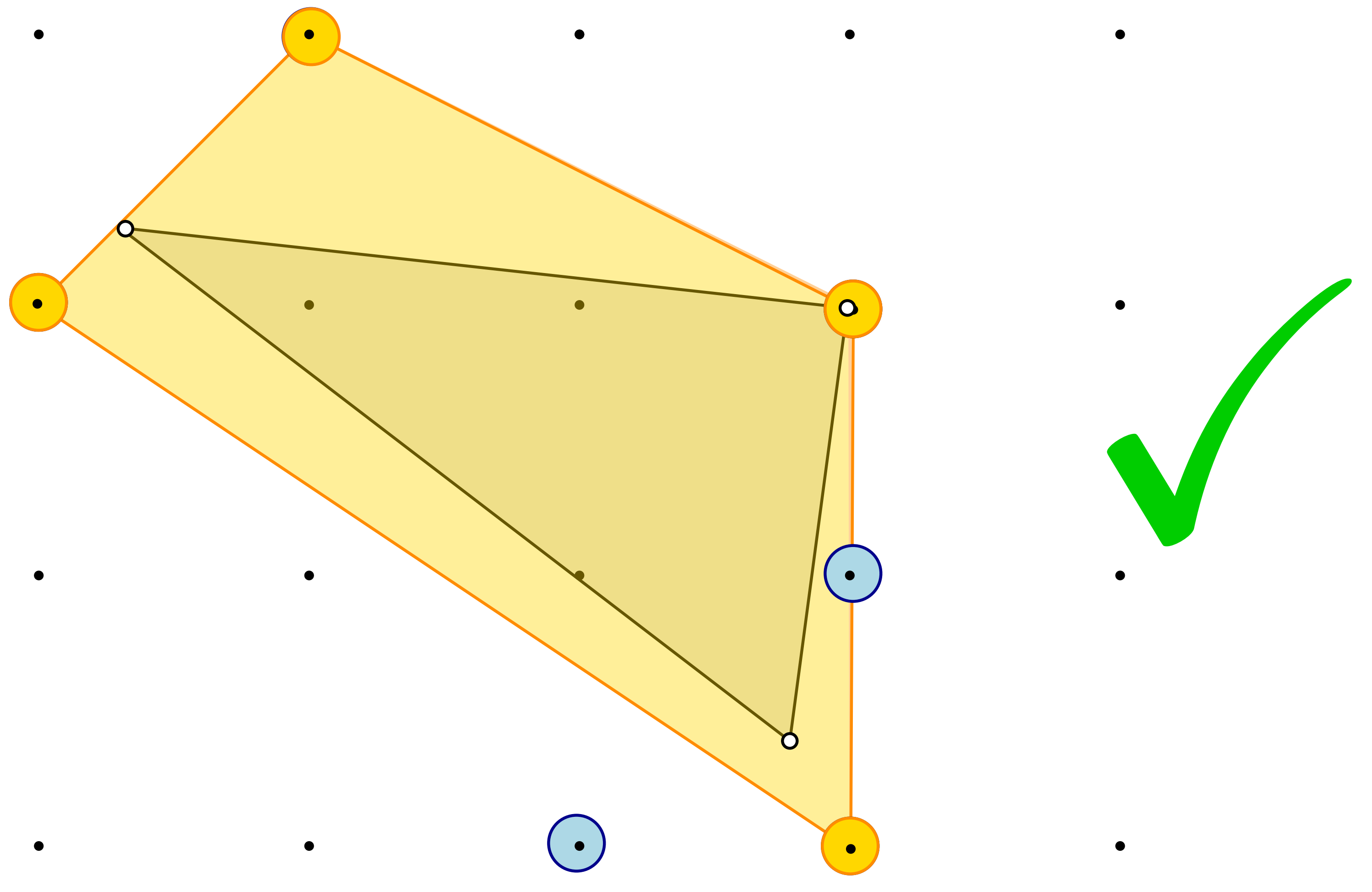
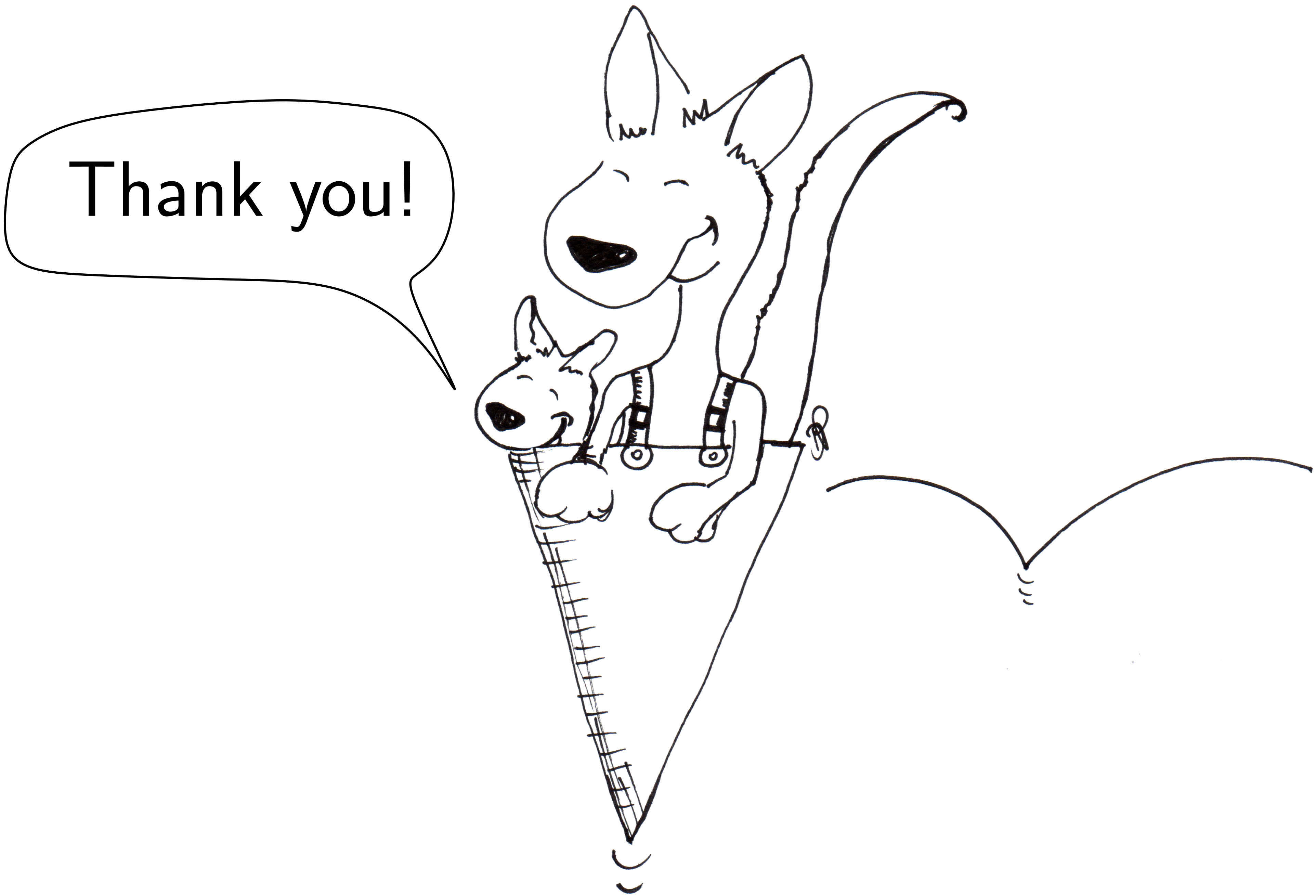


Partial Fourier-Motzkin Elimination



Demo

```
                                                    \.....|
                    Normaliz 3.0.0                   \....|
                                                      \...|
     (C) The Normaliz Team, University of Osnabrueck   \..|
                    September 2015                      \.|
                                                         \|
************************************************************
Compute: DefaultMode
Computing extreme rays as support hyperplanes of the dual cone:
************************************************************
starting primal algorithm (only support hyperplanes) ...
Generators sorted lexicographically
Start simplex 1 2 3 4 5 6 7 8 9 10
Checking for pointed ... done.
Select extreme rays via comparison ... done.
------------------------------------------------------------
************************************************************
starting primal algorithm with full triangulation ...
Roughness 7
Generators sorted by degree and lexicographically
Generators per degree:
2: 2  4: 2  14: 6
Start simplex 1 2 3 4 5 6 7 8 9 10
Pointed since graded
evaluating 1 simplices
||||||||||||||||||||||||||||||||||||||||||||||||||
1 simplices, 0 HB candidates accumulated.
1 large simplices stored
Evaluating 1 large simplices
Large simplex 1 / 1
simplex volume 416728074151872
**************************************************
```


Thank you!