Symmetry Handling in Binary Programs via Polyhedral Methods

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Discrete Optimization

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Motivation

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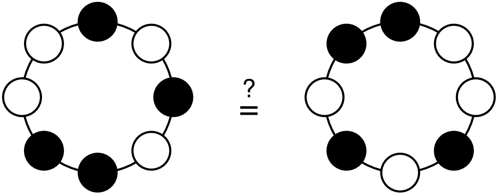
- Consider a necklace with n black (0) and white (1) beads.
- Associate with a necklace a vector in $\{0, 1\}^n$.
- necklaces $x, x' \in \{0, 1\}^n$ are "equal" $\Leftrightarrow \exists$ cyclic shift γ such that $\gamma(x) = x'$.



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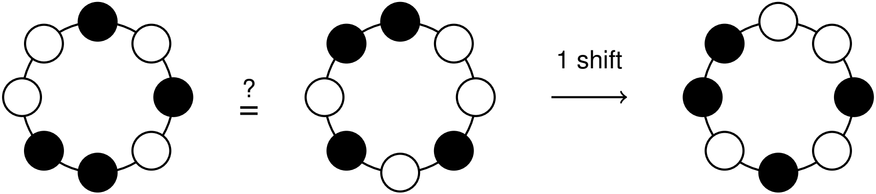

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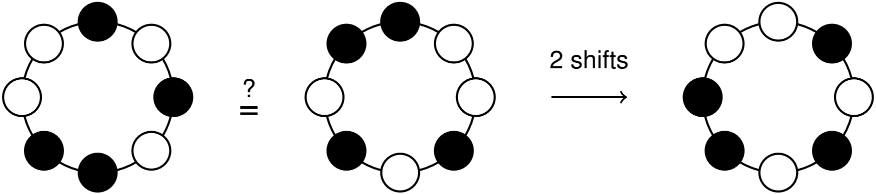

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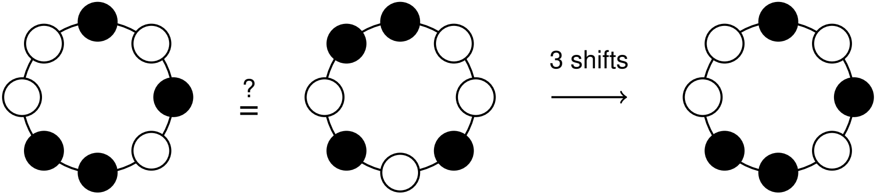

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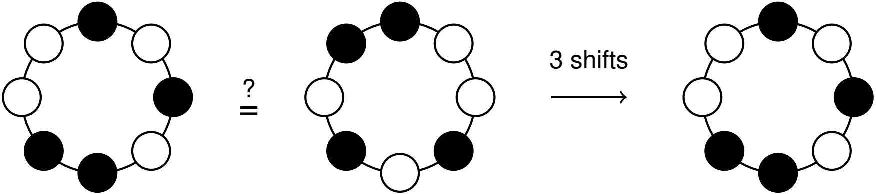

Question: How can we characterize (the convex hull of) a representative system of "distinct" necklaces?

Outline

Symmetry Detection

Representative Systems

Symretopes and Symresacks

Numerical Experiments

Symmetry Detection

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Consider binary program (BP)

$$\max\{c^\top x \ : \ Ax \leq b, \ x \in \{0,1\}^n\}.$$

We can distinguish two types of permutation symmetries of BP:

Problem Symmetry Group

Contains all permutations $\gamma \in \mathcal{S}_n$ with

- $A\gamma(x) \leq b \quad \Leftrightarrow \quad Ax \leq b$ and
- $c^\top \gamma(x) = c^\top x$.

Formulation Symmetry Group

Contains all permutations $\gamma \in \mathcal{S}_n$ for which there is $\sigma \in \mathcal{S}_m$ s.t.

- $\gamma(c) = c$,
- $\sigma(b) = b$, and
- $A_{\sigma(i),\gamma(j)} = A_{i,j}$.

Symmetry Detection

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- NP-hard

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- $\gamma(c) = c$,
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- $A_{\sigma(i),\gamma(j)} = A_{i,j}$.
- Graph-Isomorphism-hard

How to Determine Formulation Symmetries?

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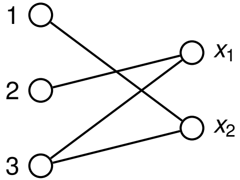
Build an auxillary colored graph and determine its color preserving automorphisms.

$$\begin{aligned}
\max\ & x_1 + x_2 & \\
& x_2 & \leq 1 \\
& x_1 & \leq 1 \\
& 2x_1 + 2x_2 & \leq 2
\end{aligned}$$



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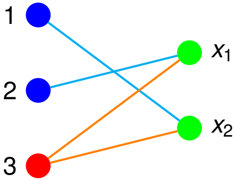

Discrete Optimization

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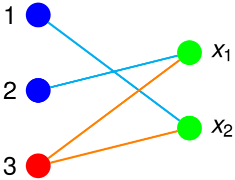

Automorphism group can be determined, e.g., with

- `bliss`,
- `nauty`, and
- `saucy`.

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Orbit Representative Systems

Given a symmetry group Γ of a binary program (BP), how can we determine a representative system of the orbits

$$\Gamma(x) = \{\gamma(x) \ : \ \gamma \in \Gamma\},$$

where x is feasible for BP?

Idea: Add

$$a^{\gamma\top} x := \sum_{i=1}^{n} (2^{n-\gamma(i)} - 2^{n-i})\, x_i \leq 0$$

for each $\gamma \in \Gamma$ to BP.

Pros and Cons

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Pros

- ${a^{\gamma}}^{\top} x \leq 0$ cuts off symmetric solutions
- $\{x \in \{0,1\}^n \; : \; Ax \leq b\} \cap \bigcap_{\gamma \in \Gamma} \{x \in \mathbb{R}^n \; : \; {a^{\gamma}}^{\top} x \leq 0\}$ is lexmax representative system of BP, see [Friedman, 2007]

Cons

- coefficients of ${a^{\gamma}}^{\top} x \leq 0$ grow exponentially large
- possibly many inequalities, one for each $\gamma \in \Gamma$

Outline

Symretopes

Definition

Given $\Gamma \leq \mathcal{S}_n$, the symretope w.r.t. Γ is the polytope

$$S(\Gamma) \coloneqq \operatorname{conv}\left(\left\{x \in \{0,1\}^n \;:\; {a^{\gamma}}^\top x \leq 0 \;\forall \gamma \in \Gamma\right\}\right).$$

Symretopes

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- If Γ is given by generators, the optimization problem over symresacks is NP-hard.
- There is $\Gamma \leq \mathcal{S}_n$ such that the coefficients of facet inequalities grow exponentially.

To avoid exponential coefficients, find tractable IP-formulation for symretopes with small coefficients.

Idea: Consider 0/1-knapsack polytope induced by ${a^{\gamma}}^\top x \leq 0$.

Symresacks

Definition

Given $\gamma \in \mathcal{S}_n$, the symresack w.r.t. γ is the polytope

$$P_\gamma \coloneqq \operatorname{conv}\left(\{x \in \{0,1\}^n \ : \ {a^\gamma}^\top x \leq 0\}\right).$$

Properties

- non-standard knapsack polytopes,
- feasible points can be completely characterized by minimal cover inequalities and box constraints,
- in general, there are exponential many minimal covers

Separation Complexity

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Theorem

The separation problem of minimal cover inequalities for P_γ and $\bar{x} \in \mathbb{R}^n$ can be solved in time $\mathcal{O}(n^2)$.

Consequences: Symmetry handling is possible with $\{0, \pm 1\}$-inequalities:

- separate minimal cover inequalities of P_γ instead of adding $a^{\gamma\top} x \leq 0$,
- separation is possible in time $\mathcal{O}(|\Gamma| n^2)$,
- avoid exponential coefficients.

Propagation Problem

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Input: Upper and lower bounds $u, l \in \{0, 1\}^n$, $\gamma \in \mathcal{S}_n$

Questions:

- Is there a vertex x of P_γ with $l \leq x \leq u$?
- Can we tighten some bounds?

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x	1	2	3	4	5	6
$\gamma(x)$	4	3	5	1	6	2

l	x_i	u
1	x_1	1
0	x_2	0
1	x_3	1
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Theorem

The propagation problem for P_γ can be solved in time $\mathcal{O}(n)$.

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Implementation

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- implemented constraint handler for symresacks in `SCIP 3.2.1` containing
  - separation of minimal cover inequalities,
  - propagation rules,
  - resolution methods if propagation detected infeasibility,
- implemented constraint handler for orbisacks [Kaibel and Loos, 2011] which are symresacks for $\gamma = (1,2)(3,4)\dots(2n-1,2n)$
- use automated symmetry detection methods of [Pfetsch and Rehn, 2015],
- use `CPLEX 12.6.1` as LP-solver

Numerical Experiments

tests were run on

- Linux cluster with Intel i3 3.2GHz dual core processors,
- 8GB memory,
- single thread
- time limit: 3600 seconds

test instances

- MIPLIB 2010 benchmark
- SONET network design [Sherali and Smith, 2001]
- Wagon Load Balancing [Ghoniem and Sherali, 2011]

Numerical Experiments: MIPLIB 2010 benchmark

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number of solved instances, default 60/87

	prop	sepa	sepa+prob
orbisacks	61	62	61
symresacks	62	60	61

speed up of solution time

	prop	sepa	sepa+prob
orbisacks	0.94	0.91	0.93
symresacks	0.93	1.00	0.97

reduction of nodes

	prop	sepa	sepa+prob
orbisacks	0.93	0.87	0.82
symresacks	0.91	0.93	0.92

Numerical Experiments: SONET

number of solved instances, default 50/50

	prop	sepa	sepa+prob
orbisacks	50	50	50
symresacks	50	50	50

speed up of solution time

orbisacks	0.06	0.04	0.04
symresacks	0.06	0.04	0.04

reduction of nodes

orbisacks	0.04	0.02	0.02
symresacks	0.04	0.02	0.02

Numerical Experiments: Wagon Load Balancing

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number of solved instances, default 2/120

	prop	sepa	sepa+prob
orbisacks	2	2	2
symresacks	10	120	120

speed up of solution time

orbisacks	1.01	1.03	1.01
symresacks	1.00	0.02	0.02

reduction of nodes

orbisacks	1.00	1.04	1.00
symresacks	0.95	0.01	0.01

Thank you for your attention!

Literature

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