The Computational Complexity of Spark, RIP, and NSP

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Sparse Recovery Conditions

- $\min\{\ \|x\|_0\ :\ Ax = b\ \}$ is NP-hard (also with constraint $\|Ax - b\|_2 \leq \varepsilon$)
- various conditions for k-sparse solution uniqueness and recoverability by heuristics such as OMP or ℓ_1-minimization

Complexity Rumors ...

Spark, RIP, and NSP are very often mentioned to be intractable / NP-hard,
but apparently no proof or reference anywhere in CS literature!

- In particular, hardness often “explained” solely by “combinatorial nature”

  (this reasoning is false – many combinatorial problems are in P)

Outline

1 Computational Complexity Basics

2 Confirming the Intractability Rumors ...
- Spark
- Restricted Isometry Property (RIP)
- Nullspace Property (NSP)

3 Conclusions

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P, NP, and coNP – Hardness and completeness (informally...)

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- P : deterministic-polynomial-time solvable (decision) problems
- NP : nondeterministic-polynomial-time solvable (decision) problems
  - polynomial certificate for “yes” answers, but no poly.-time solution algorithm (unless P=NP)
- coNP : complementary class of NP
  - polynomial certificate for “no” answers, but no poly.-time solution algorithm (unless P=NP)

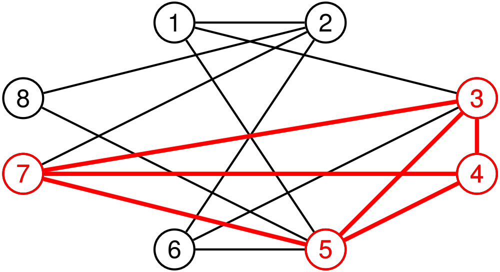
- coNP-hard ≡ NP-hard : (decision or optimization) problems for which existence of a polynomial solution algorithm would imply P=NP

- (co)NP-complete : NP-hard (decision) problems contained in (co)NP

Examples

- LINEAR PROGRAMMING $\in$ P.
- A classical NP-complete problem: the k-CLIQUE problem
  Given a graph G and a positive integer k, does G contain a clique of size k?

  Example: 4-clique $\{3, 4, 5, 7\}$.



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The Spark of a Matrix

Definition

$$\mathrm{spark}(A) \;\coloneqq\; \min \; \|x\|_0 \quad \text{s.t.} \quad Ax = 0,\; x \neq 0$$

- why care?
  - unique k-sparse ℓ_0-solution if and only if $k < \mathrm{spark}(A)/2$
- a.k.a. *girth* of the vector matroid $\mathcal{M}(A)$ on A:

$$\mathrm{spark}(A) = \min\{\, |C| \;:\; C \text{ circuit of } \mathcal{M}(A) \,\},$$

  circuit: inclusion-wise minimal collection of linearly dependent columns
- for graphic matroids: polynomial time; for transversal matroids: NP-hard

Spark Complexity – An overlooked early result

- Khachiyan, 1995:

  Given $A \in \mathbb{Q}^{m \times n}$, it is NP-complete to decide whether A has an $(m \times m)$-submatrix with zero determinant.

  $$\Updownarrow$$

  It is NP-complete to decide whether $\operatorname{spark}(A) \leq m$.

- Observation: *"Is A full-spark?"* (*"*$\operatorname{spark}(A) = m + 1$*?") is coNP-complete.*

  (previously only known to be "hard for NP under *randomized* reductions", based on probabilistic matrix representation of transversal matroids [Alexeev et al.])

Spark Complexity – New Result

Theorem 1 (T. & Pfetsch)

Given a matrix $A \in \mathbb{Q}^{m \times n}$ (with rank(A) = $m < n$) and a positive integer $k < m$, it is NP-complete to decide whether spark(A) $\leq k$ (or spark(A) = k).

- Difference to Khachiyan's result: $k < m$ with *full (row)-rank A*
  (Khachiyan's proof extends to $k < m$ only by appending zero-rows)

Corollary

Given a matrix A, computing spark(A) is NP-hard.

(Polyn. algo. to compute spark(A) *could decide "*spark(A) $\leq k$*?" in poly-time.* $\square$*)*

Proof Sketch for Theorem 1

Reduction from k-CLIQUE:

- given instance: $G = (V, E)$ and $k \in \mathbb{N}$ (wlog $k > 4$), with $n := |V|$ and $m := |E|$
- construct a matrix A of size $(n + \binom{k}{2} - k - 1) \times m$
  - first n rows: set $a_{ie} = 1$ iff $i \in e$, and 0 else *(incidence matrix of G)*
  - remaining rows ($n + i$ for $i = 1, \ldots, \binom{k}{2} - k - 1$): set $a_{(n+i)e} = (U + i + 1)^{e-1}$ *(sub-Vandermonde matrix)*
- G has a k-clique if and only if $\operatorname{spark}(A) \leq \binom{k}{2}$ (in fact, $\operatorname{spark}(A) = \binom{k}{2}$).
  - a specific choice of U [cf. Chistov et al.] and some technical auxiliary results on graphs and incidence matrices yield the desired linear (in)dependency properties.
- containment in NP: "guess" x with $Ax = 0$ ($\Rightarrow$ can assume $x \in \mathbb{Q}^n$); can verify $Ax = 0$, $\|x\|_0 = \binom{k}{2}$, and that $\operatorname{supp}(x)$ is a circuit in poly-time.

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The Restricted Isometry Property (RIP)

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Definition

A matrix $A \in \mathbb{R}^{m \times n}$ satisfies the RIP of order k with constant δ_k if

$$(1 - \delta_k)\|x\|_2^2 \;\le\; \|Ax\|_2^2 \;\le\; (1 + \delta_k)\|x\|_2^2 \qquad \forall\, x : \; \|x\|_0 \le k. \qquad (k, \delta_k)\text{-RIP}$$

Restricted Isometry Constant (RIC): $\underline{\delta}_k \coloneqq \min\{\, \delta_k \;:\; A \text{ satisfies } (k, \delta_k)\text{-RIP} \,\}$

why care?

- ℓ_o-ℓ_1-equivalence for k-sparse solutions if $\underline{\delta}_{2k} < \sqrt{2} - 1$ [Candès, 2008], or if $\underline{\delta}_k < 0.307$ [Cai, Wang & Xu, 2010], ...
- certain random matrices have desirable RIP with high probability

Central RIP-related Complexity Issues

- RIC computation: Is it hard to *compute* the RIC $\underline{\delta}_k$ (given A and k)?
- RIP certification: Is it hard to *decide* whether $\underline{\delta}_k < \delta$ (given A, k, δ)?

Complexity of RIP Certification

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Theorem 2 (RIP Certification I) (T. & Pfetsch)

Given a matrix $A \in \mathbb{Q}^{m \times n}$ and a positive integer k, deciding whether there exists some constant $\delta_k < 1$ such that A satisfies the (k, δ_k)-RIP is coNP-complete.

Theorem 3 (RIP Certification II) (T. & Pfetsch)*

Given a matrix $A \in \mathbb{Q}^{m \times n}$, a positive integer k, and some constant $\delta_k \in (0, 1)$, deciding whether A satisfies the (k, δ_k)-RIP is (co)NP-hard.

* independently obtained by [Bandeira et al.], using Khachiyan's spark result (i.e., $k = m$).

Complexity of RIC Computation

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Corollary

Computation of the RIC $\underline{\delta}_k$ is NP-hard.

Proof: A polynomial algorithm to compute the RIC could be used to decide RIP CERTIFICATION (I or II) in polynomial time. $\square$

A Useful Lemma

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Lemma 1

Let $A = (a_{ij}) \in \mathbb{Q}^{m\times n}$ and define $\alpha \coloneqq \max |a_{ij}|$, $C \coloneqq 2^{\lceil \log_2(\alpha\sqrt{mn})\rceil}$, and $\tilde{A} \coloneqq \frac{1}{C}A$. Then

$$\|\tilde{A}x\|_2^2 \leq (1+\delta)\|x\|_2^2 \qquad \text{for all } x \in \mathbb{R}^n \text{ and } \delta \geq 0.$$

Why useful?

(k, δ_k)-RIP for $\tilde{A}$ reduces to "$(1-\delta_k)\|x\|_2^2 \leq \|\tilde{A}x\|_2^2 \;\; \forall$ k-sparse x", i.e.,
only the lower RIP inequality is relevant!

Proof of Theorem 2 (RIP CERTIFICATION I) ("(k, δ_k)-RIP for some $\delta_k < 1$"?)

Reduction from SPARK ("spark(A) $\leq k$?"):

- instance for RIP-problem: $\tilde{A} = \frac{1}{C}A$, k (note $\mathcal{M}(A) = \mathcal{M}(\tilde{A})$)
- If spark(A) $\leq k$, there exists k-sparse $x \neq 0$ with $\tilde{A}x = 0$. Then

$$(1-\delta_k)\|x\|_2^2 \leq \|\tilde{A}x\|_2^2 = 0 \qquad \Rightarrow \qquad \delta_k \geq 1.$$

- Conversely, suppose there is no $\delta_k < 1$ s.t. $\tilde{A}$ is (k, δ_k)-RIP. Then

$$\exists\, x \text{ with } 1 \leq \|x\|_0 \leq k \qquad \text{s.t.} \qquad 0 \geq (1-\delta_k)\|x\|_2^2 = \|\tilde{A}x\|_2^2 \geq 0,$$

  hence $\tilde{A}x = 0$. $\Rightarrow \exists$ *circuit* ($\subseteq$supp(x)) of size at most k, thus spark(A) $\leq k$.

- RIP CERTIFICATION I $\in$ coNP: certificate is x with $1 \leq \|x\|_0 \leq k$ which tigthly satisfies the $(k, 1)$-RIP; implies $Ax = 0$ (so can assume $x \in \mathbb{Q}^n$). $\square$

Another Useful Lemma

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Lemma 2

Given a matrix $A \in \mathbb{Q}^{m \times n}$ and a positive integer $k \leq n$, if spark(A) $> k$, there exists a rational constant $\varepsilon > 0$ such that

$$\|Ax\|_2^2 \geq \varepsilon \, \|x\|_2^2 \qquad \text{for all } x \text{ with } 1 \leq \|x\|_0 \leq k.$$

Why useful?

reveals a "rationality gap": $\delta_k < 1 \quad \Leftrightarrow \quad \delta_k \leq 1 - \varepsilon$

Complexity of RIP Certification II
("(k, δ_k)-RIP with $\delta_k \leq \delta$ for *given* $\delta \in (0,1)$"?)

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Proof Sketch:

- essentially extend the proof of Theorem 2 by means of previous Lemma:

$$\mathrm{spark}(A) \leq k \quad \Leftrightarrow \quad \tilde{A} \text{ not } (k, \delta_k)\text{-RIP with some } \delta_k < 1 \qquad \text{(Theorem 2)}$$

$$\Leftrightarrow \quad \tilde{A} \text{ not } (k, 1-\varepsilon)\text{-RIP} \qquad \text{(Theorem 3)}$$

Remark: Containment in coNP not known.

(rationality of the certificate x is not obvious, since no longer $Ax = 0$)

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The Nullspace Property (NSP)

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Definition

A matrix $A \in \mathbb{R}^{m \times n}$ satisfies the NSP of order k with constant α_k if

$$\|x\|_{k,1} \coloneqq \max_{S:|S|=k} \sum_{i \in S} |x_i| \leq \alpha_k \|x\|_1 \qquad \forall\, x : \; Ax = 0. \qquad\qquad (k, \alpha_k)\text{-NSP}$$

Nullspace Constant (NSC): $\underline{\alpha}_k \coloneqq \min\{\, \alpha_k \;:\; A \text{ satisfies } (k, \alpha_k)\text{-NSP} \,\}$

why care?

- ℓ_0-ℓ_1-equivalence *if and only if* $\underline{\alpha}_k < 1/2$

Complexity of the NSP

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Theorem 4 (T. & Pfetsch)

Given a matrix $A \in \mathbb{Q}^{m \times n}$ and a positive integer k, deciding whether there exists some constant $\alpha_k < 1$ such that A satisfies the (k, α_k)-NSP is coNP-complete.

Corollary

Computation of the NSC $\underline{\alpha}_k$ is NP-hard.

Proof of Theorem 4

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Reduction from SPARK ("spark(A) $\leq k$?"):

- ▷ instance for NSP-decision problem: A, k
- ▷ If spark(A) $\leq k$, there exists x with $Ax = 0$ and $1 \leq \|x\|_0 \leq k$. Then, $\|x\|_{k,1} = \|x\|_1$, and therefore $\alpha_k \geq 1$ (in fact, $\alpha_k = 1$).
- ▷ Conversely, suppose there is no $\alpha_k < 1$ s.t. A satisfies the (k, α_k)-NSP. Then there is some x with $Ax = 0$ and $1 \leq \|x\|_0 \leq k$ such that $\|x\|_{k,1} = \|x\|_1$ (otherwise $\alpha_k < 1$ *was* possible).
  $\Rightarrow \exists$ circuit ($\subseteq$supp(x)) of size at most k, whence spark(A) $\leq k$.
- ▷ $\in$coNP: $\alpha_k \leq 1$ (trivially) $\Rightarrow$ "no"-certificate is a k-sparse $x \in \mathbb{Q}^n$ s.t. $Ax = 0$ and $\|x\|_{k,1} \leq \alpha_k \|x\|_1$ with $\alpha_k = 1$ is tightly satisfied, i.e., $\|x\|_{k,1} = \|x\|_1$ $\square$

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Concluding Remarks

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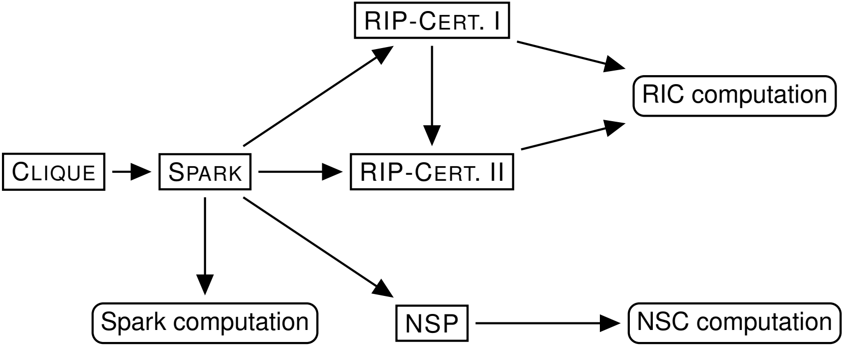
- Suspicions confirmed: Spark, RIP, and NSP are all NP-hard indeed



Concluding Remarks

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- Suspicions confirmed: Spark, RIP, and NSP are all NP-hard indeed

- Existing approximation/relaxation algorithms well justified
- More work on *exact* algorithms desirable
  - NP-hardness means *not all* instances can be solved efficiently – existence of *practically* efficient methods not necessarily excluded!

- Still open: Complexity of verifying (e.g.) $\underline{\delta}_k < 0.307$, $\underline{\alpha}_k < 1/2$, ... ? Complexity of approximating $\underline{\delta}_k$ or $\underline{\alpha}_k$?

- Details, and more results, in our paper

  arXiv: 1205.2081 (new version v4!)