Projection onto the k-Cosparse Set is NP-hard

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Cosparse Analysis Model

Typical sparse recovery approach: For $A \in \mathbb{R}^{m\times n}$, $b \in \mathbb{R}^m$ $(m < n)$, solve

$$\min_{\mathbf{x}\in\mathbb{R}^n} \|\mathbf{x}\|_0 \quad \text{s.t.} \quad \mathbf{Ax} = \mathbf{b} \qquad (\text{or } \|Ax - b\|_2 \le \delta). \qquad (\mathrm{P}_0)$$

Alternative cosparse analysis model:
Assume Ωx is sparse, where Ω is an analysis operator:

$$\min_{\mathbf{x}\in\mathbb{R}^n} \|\mathbf{\Omega x}\|_0 \quad \text{s.t.} \quad \mathbf{Ax} = \mathbf{b} \qquad (\mathrm{C}_0)$$

Iterative Algorithms

Both (P_0) and (C_0) are NP-hard $\rightsquigarrow$ employ heuristic solution methods

Popular approach for (P_0): Iterative Hard-Thresholding (IHT)

Alternate

- gradient descent step to reduce error $\|Ax^k - b\|_2$
- projection onto $\{x : \|x\|_0 \le k\}$, i.e., hard-thresholding (trivial: keep only the k largest entries)

IHT- and Greedy-like methods for (C_0) require projection onto set of k-cosparse vectors, $\{x : \|\Omega x\|_0 \le k\}$

Computational Complexity of k-Cosparse Projection Problems

Theorem

Given $\Omega \in \mathbb{R}^{r\times n}$ $(r > n)$, $\omega \in \mathbb{R}^n$, and a positive integer $k \in \mathbb{N}$, it is NP-hard in the strong sense to solve the k-cosparse ℓ_p-norm projection problem

$$\min_{\mathbf{z}\in\mathbb{R}^n} \|\omega - \mathbf{z}\|_p^q \quad \text{s.t.} \quad \|\mathbf{\Omega z}\|_0 \le \mathbf{k}, \qquad (k\text{-CoSP}_p)$$

for any $p \in \mathbb{N} \cup \{\infty\}$, $p > 1$, where $q = p$ if $p < \infty$ and $q = 1$ if $p = \infty$. The problem remains strongly NP-hard even if ω has only binary coefficients in $\{0, 1\}$ (with exactly one entry nonzero) and Ω has only ternary or bipolar coefficients in $\{-1, 0, +1\}$ or $\{-1, 1\}$, respectively.

Proof idea. Reduction from $\mathrm{MinULR}_0^=(A, K)$: Given a matrix $A \in \mathbb{Q}^{r\times n}$ and a positive integer $K \in \mathbb{N}$, decide whether there exists a vector $z \in \mathbb{R}^n$ such that $z \neq 0$ and at most K of the r equalities in the system $Az = 0$ are violated. Known to be strongly NP-complete even for ternary or bipolar A (Amaldi & Kann, 1995).

Given an instance (A, K) of $\mathrm{MinULR}_0^=$ (w.l.o.g. $r > n$), we reduce it to n instances of $(k\text{-CoSP}_p)$:
For all $i = 1, \ldots, n$, define a k-cosparse projection instance by $\Omega = A$, $\omega = e_i$ and $k = K$ (where e_i denotes the i-th unit vector). Observe that since $z = 0$ is always a feasible point, $f_{i,k}^{(p)} := \min_{z\in\mathbb{R}^n}\{\, \|e_i - z\|_p^q : \|\Omega z\|_0 \le k \,\} \le 1$ for all i, hence $f_k^{(p)} := \min_{1\le i\le n} f_{i,k}^{(p)} \le 1$.

$\rightsquigarrow$ Claim: $f_k^{(p)} < 1$ if and only if $\mathrm{MinULR}_0^=(A, K)$ has a positive answer. (Verifying the claim proves the Theorem. $\square$)

Want to see the (technical) details? Ask us for the paper!

Corollary: It is strongly NP-hard to compute a minimizer for $(k\text{-CoSP}_p)$, in particular, the Euclidean projection $\Pi_{\Omega,k}(\omega) := \arg\min_{z\in\mathbb{R}^n}\{\, \|\omega - z\|_2 : \|\Omega z\|_0 \le k \,\}$.

Corollary: The associated decision problem "Does there exist some $z \in \mathbb{R}^n$ such that $\|\omega - z\|_p^q < \gamma$ and $\|\Omega z\|_0 \le k$?" is strongly NP-complete (for $p = 2, \infty$).

Consequences, Conclusions & Open Questions

Strong NP-hardness $\Longrightarrow$ no Fully Polynomial-Time Approximation Scheme (FPTAS), no Pseudo-Polynomial Exact Algorithm (unless P=NP)!

(FPTAS: given $\varepsilon > 0$, finds value $\le (1+\varepsilon)\cdot$min.-value in time polynomially depending on input encoding length and $1/\varepsilon$)
(Pseudo-Poly. Alg.: finds exact optimum in time polynomial in the *numeric* value of the input, but not its *encoding length*)

$\rightsquigarrow$ existence of other approximation algorithms? (may still be useful in practice despite bad theoretical running time bounds, i.e., at least exponential in $1/\varepsilon$)

Easier special cases: Ω unitary $\Rightarrow$ hard-tresholding achieves k-cosparse projection w.r.t. any ℓ_p-norm;
Ω 1D finite difference or fused Lasso operator $\Rightarrow$ Euclidean projection achieved using dynamic programming

Future Challenges: Complexity of $(k\text{-CoSP}_p)$ for $0 < p \le 1$?
Finding (practically) efficient approximation schemes for $(k\text{-CoSP}_p)$, or establishing further (perhaps negative) approximability results!

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