Projection onto the $k$-Cosparse Set is NP-hard

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Cosparse Analysis Model

Typical sparse recovery approach: For $A \in \mathbb{R}^{m \times n}$, $b \in \mathbb{R}^m$ ($m < n$), solve

$$\min_{x \in \mathbb{R}^n} \|x\|_0 \quad \text{s.t.} \quad Ax = b \quad \text{(or } \|Ax - b\|_2 \leq \delta). \quad (P_0)$$

Alternative cosparse analysis model:
Assume $\Omega$ is sparse, where $\Omega$ is an analysis operator:

$$\min_{x \in \mathbb{R}^n} \|\Omega x\|_0 \quad \text{s.t.} \quad Ax = b \quad (C_0)$$

Cosparse projection onto the $k$-Cosparse Set

Assume $\Omega x$ is sparse, where $\Omega$ is an analysis operator:

$$\min_{x \in \mathbb{R}^n} \|\Omega x\|_0 \quad \text{s.t.} \quad Ax = b \quad (C_0)$$

Computational Complexity of $k$-Cosparse Projection Problems

Theorem

Given $\Omega \in \mathbb{R}^{r \times n}$ ($r > n$), $\omega \in \mathbb{R}^r$, and a positive integer $k \in \mathbb{N}$, it is NP-hard in the strong sense to solve the $k$-cosparse $\ell_p$-norm projection problem

$$\min_{x \in \mathbb{R}^n} \|\omega - z\|_p \quad \text{s.t.} \quad \|\Omega z\|_0 \leq k. \quad (k\text{-CoSP}_p)$$

for any $p \in \mathbb{R} \cup \{\infty\}, p > 1$, where $q = p$ if $p < \infty$ and $q = 1$ if $p = \infty$. The problem remains strongly NP-hard even if $\omega$ has only binary coefficients in $\{0, 1\}$ (with exactly one entry nonzero) and $\Omega$ has only ternary or bipolar coefficients in $\{-1, 0, 1\}$ or $\{-1, 1\}$, respectively.

Proof idea. Reduction from MinULR$^n$$(A,K)$: Given a matrix $A \in \mathbb{Q}^{r \times n}$ and a positive integer $K \in \mathbb{N}$, decide whether there exists a vector $z \in \mathbb{R}^n$ such that $z \neq 0$ and at most $K$ of the $r$ equalities in the system $Az = 0$ are violated. Known to be strongly NP-complete even for ternary or bipolar $A$ (Amaldi & Kann, 1995).

Given an instance $(A,K)$ of MinULR$^n$$(w.l.o.g. r > n)$, we reduce it to $n$ instances of $(k\text{-CoSP}_p)$:

- For all $i = 1, \ldots, n$, define a $k$-cosparse projection instance by $\Omega = A, \omega = e_i$ and $k = K$ (where $e_i$ denotes the $i$-th unit vector).

Observe that since $z = 0$ is always a feasible point, $r(i) = \min_{x \in \mathbb{R}^n} \{e_i - z : ||\Omega z||_0 \leq k\} \leq 1$ for all $i$, hence $r(i) = \min_{x \in \mathbb{R}^n} r(i) \leq 1$.

Claim: $r(i) < 1$ if and only if $\text{MinULR}^n(A,K)$ has a positive answer. (Verifying the claim proves the Theorem. $\square$)

Consequences, Conclusions & Open Questions

Strong NP-hardness $\Rightarrow$ no Fully Polynomial-Time Approximation Scheme (FPTAS), no Pseudo-Polynomial Exact Algorithm (unless P=NP)!

(FPTAS: given $\epsilon > 0$, finds value $\leq (1 + \epsilon) \text{-min. value in time polynomially depending on input encoding length and } 1/\epsilon$)
(Pseudo-Poly. Alg.: finds exact optimum in time polynomial in the numeric value of the input, but not its encoding length)

$\Rightarrow$ existence of other approximation algorithms? (may still be useful in practice despite bad theoretical running time bounds, i.e., at least exponential in $1/\epsilon$)

Easier special cases: $\Omega$ unitary $\Rightarrow$ hard-thresholding achieves $k$-cosparse projection w.r.t. any $\ell_p$-norm:

$\Omega$ 1D finite difference or fused Lasso operator $\Rightarrow$ Euclidean projection achieved using dynamic programming

Future Challenges: Complexity of $(k\text{-CoSP}_p)$ for $0 < p \leq 1$?
Finding (practically) efficient approximation schemes for $(k\text{-CoSP}_p)$, or establishing further (perhaps negative) approximability results!