

## Introduction

This is an exact controllability problem for the one-dimensional wave equation by Dirichlet boundary actuation. Such problems were thoroughly analyzed in [Gugat et al. \[2005\]](#), where the present problem appears as Example 1.

## Variables & Notation

### Unknowns

$$\begin{aligned} f_1, f_2 \in L^\infty(0, T) & \quad \text{control variables} \\ y & \quad \text{state variable} \end{aligned}$$

### Given Data

The given data is chosen in a way which admits an analytic solution.

$$\begin{aligned} y_0(x) = x - 1/2 & \quad \text{target displacement} \\ y_1(x) = 1 & \quad \text{target velocity} \\ T = 3.25 & \quad \text{final time} \\ \Omega = (0, 1) & \quad \text{computational domain} \\ \hat{r} = 5/28 & \quad \text{auxiliary} \\ g_1(t) = -1/4 & \quad \text{auxiliary function} \\ g_2(t) = 3/4 - t & \quad \text{auxiliary function.} \end{aligned}$$

## Problem Description

$$\begin{aligned} \text{Minimize} \quad & \max \{ \|f_1\|_{L^\infty(0,T)}, \|f_2\|_{L^\infty(0,T)} \} \\ \text{s.t.} \quad & \begin{cases} y_{tt} = y_{xx} & \text{in } \Omega \times (0, T) \\ y(x, 0) = 0 & \text{in } \Omega & y_t(x, 0) = 0 & \text{in } \Omega \\ y(x, T) = y_0(x) & \text{in } \Omega & y_t(x, T) = y_1(x) & \text{in } \Omega \\ y(0, t) = f_1(t) & \text{in } (0, T) & y(1, t) = f_2(t) & \text{in } (0, T). \end{cases} \end{aligned}$$

## Optimality System

An optimality system is not provided in [Gugat et al. \[2005\]](#) but rather the exact solution is constructed analytically.

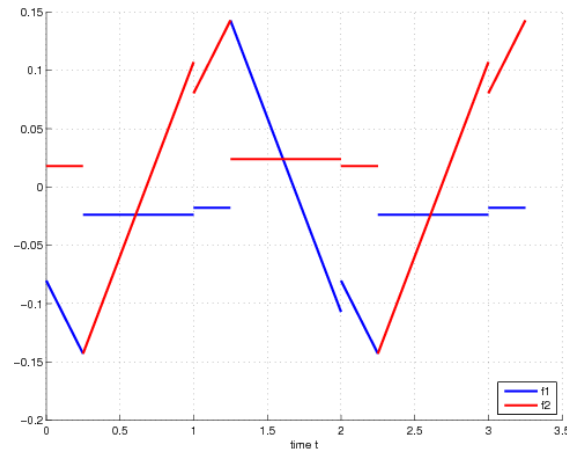


Figure 0.1: optimal controls

## Supplementary Material

An optimal control can be derived analytically from [Gugat et al., 2005, Theorem 2.2]:

$$f_1(T-t) = \begin{cases} \frac{g_1(t)+\hat{r}}{4}, & t \in [0, 0.25] \\ \frac{g_1(t)+\hat{r}}{3}, & t \in [0.25, 1] \\ \frac{-g_2(t-1)+\hat{r}}{4}, & t \in [1, 1.25] \\ \frac{-g_2(t-1)+\hat{r}}{3}, & t \in [1.25, 2] \\ \frac{g_1(t-2)+\hat{r}}{4}, & t \in [2, 2.25] \\ \frac{g_1(t-2)+\hat{r}}{3}, & t \in [2.25, 3] \\ \frac{-g_2(t-3)+\hat{r}}{4}, & t \in [3, 3.25] \end{cases} \quad f_2(T-t) = \begin{cases} \frac{g_2(t)-\hat{r}}{4}, & t \in [0, 0.25] \\ \frac{g_2(t)-\hat{r}}{3}, & t \in [0.25, 1] \\ \frac{-g_1(t-1)-\hat{r}}{4}, & t \in [1, 1.25] \\ \frac{-g_1(t-1)-\hat{r}}{3}, & t \in [1.25, 2] \\ \frac{g_2(t-2)-\hat{r}}{4}, & t \in [2, 2.25] \\ \frac{g_2(t-2)-\hat{r}}{3}, & t \in [2.25, 3] \\ \frac{-g_1(t-3)-\hat{r}}{4}, & t \in [3, 3.25]. \end{cases}$$

This pair of optimal controls may not be unique, but it is the unique one with minimal  $L^2(0, T)$  norm. The corresponding displacement  $y$  is piecewise linear, and the optimal velocity  $y_t$  is piecewise constant on areas bounded by characteristic curves of the equation  $y_{tt} = y_{xx}$ . Figure 0.1 displays the optimal controls. A plot of the optimal displacement is provided in [Gugat et al., 2005, Figure 2.1].

## References

- M. Gugat, G. Leugering, and G. Sklyar.  $L^p$ -optimal boundary control for the wave equation. *SIAM Journal on Control and Optimization*, 44(1):49–74, 2005. ISSN 0363-0129. doi: [10.1137/S0363012903419212](https://doi.org/10.1137/S0363012903419212).