## Introduction

This is a distributed optimal control problem for a semilinear 1D parabolic reaction-diffusion equation, where traveling wave fronts occur. The state equation is known as *Schlögl model* in physics and as *Nagumo equation* in neurobiology. In this context, various goals of optimization are of interest, for instance the stopping, acceleration, or extinction of a traveling wave. Here, we discuss the problem of stopping a wave front at a certain time and keeping it fixed afterwards. This problem appears in [Buchholz et al., 2013, Section 5.4]. In the same paper, additional examples can be found which cover the other optimization goals mentioned above. It has the explicitly known optimal control (forcing) $f_{\text{stop}}$ defined below and displayed in Figure 0.2.

## Variables & Notation

### Unknowns

$$f \in L^2(Q) \quad \text{control variable (forcing)}$$

$$u \in L^2(0,T;H^1(\Omega)) \cap H^1(0,T;H^1(\Omega)') \cap L^\infty(Q) \quad \text{state variable}$$

### Given Data

| | |
|---|---|
| $\Omega = (0, L)$ | spatial domain |
| $L = 20$ | side length of domain |
| $Q = \Omega \times (0, T)$ | computational domain |
| $T = 5$ | terminal time |
| $u_0(x) = \begin{cases} 1.2\sqrt{3}, & x \in [9, 11] \\ 0, & \text{elsewhere,} \end{cases}$ | initial condition |
| $\lambda = 10^{-6}$ | Tikhonov regularization parameter |
| $u_Q(\cdot, t) = \begin{cases} u_{\text{nat}}(\cdot, t), & t \in [0, 2.5] \\ u_{\text{nat}}(\cdot, 2.5), & t \in (2.5, T] \end{cases}$ | desired state |
| $u_{\text{nat}}$ | solution of the PDE (0.1) for $f \equiv 0$. |

The natural uncontrolled state $u_{\text{nat}}$ is shown in Figure 0.1. In the figure, the horizontal axis shows the spatial variable $x$ while the vertical one displays the time $t$. An analytical expression for $u_{\text{nat}}$ is not known.

## Problem Description

$$\text{Minimize} \quad \frac{1}{2}\iint_Q (u(x,t) - u_Q(x,t))^2 \, \mathrm{d}x \, \mathrm{d}t + \frac{\lambda}{2}\iint_Q f^2(x,t) \, \mathrm{d}x \, \mathrm{d}t$$

$$\text{s.t.} \quad \begin{cases} \dfrac{\partial u}{\partial t}(x,t) - \dfrac{\partial^2 u}{\partial x^2}(x,t) + \dfrac{1}{3}u^3(x,t) - u(x,t) = f(x,t) & \text{in } Q \\ u(x,0) = u_0(x) & \text{in } \Omega \\ \dfrac{\partial u}{\partial x}(0,t) = \dfrac{\partial u}{\partial x}(L,t) = 0 & \text{in } (0,T). \end{cases} \tag{0.1}$$

Notice that the PDE has a non-monotone nonlinearity. The associated homogeneous elliptic (stationary) equation admits three different solutions; namely, the functions $u_1(x) \equiv -\sqrt{3}$, $u_2(x) \equiv 0$, and $u_3(x) \equiv \sqrt{3}$.

## Optimality System

The following optimality system for the state $u$, the control $f$, and the adjoint state $p$, given in the strong form, represents first-order necessary optimality conditions.

$$\begin{aligned} \frac{\partial u}{\partial t}(x,t) - \frac{\partial^2 u}{\partial x^2}(x,t) + \frac{1}{3}u^3(x,t) - u(x,t) &= f(x,t) && \text{in } Q \\ u(x,0) &= u_0(x) && \text{in } \Omega \\ \frac{\partial u}{\partial x}(0,t) = \frac{\partial u}{\partial x}(L,t) &= 0 && \text{in } (0,T), \\ -\frac{\partial p}{\partial t}(x,t) - \frac{\partial^2 p}{\partial x^2}(x,t) + u^2(x,t)\,p(x,t) - p(x,t) &= u(x,t) - u_Q(x,t) && \text{in } Q \\ p(x,T) &= 0 && \text{in } \Omega \\ \frac{\partial p}{\partial x}(0,t) = \frac{\partial p}{\partial x}(L,t) &= 0 && \text{in } (0,T), \\ f(x,t) &= -\frac{1}{\lambda}p(x,t) && \text{in } Q. \end{aligned}$$

## Supplementary Material

The optimal state and the optimal control are given by:

$$u(x,t) = u_Q(x,t),$$

$$f_{\text{stop}}(x,t) = \begin{cases} 0 & \text{for } t \le 2.5, \\ \frac{1}{3}\,u_{\text{nat}}^3(x,2.5) - u_{\text{nat}}(x,2.5) - \dfrac{\partial^2}{\partial x^2}u_{\text{nat}}(x,2.5), & \text{for } t > 2.5. \end{cases}$$

These functions are shown in Figure 0.2.

## References

R. Buchholz, H. Engel, E. Kammann, and F. Tröltzsch. On the optimal control of the Schlögl model. *Computational Optimization and Applications*, 56(1):153–185, 2013. doi: 10.1007/s10589-013-9550-y.

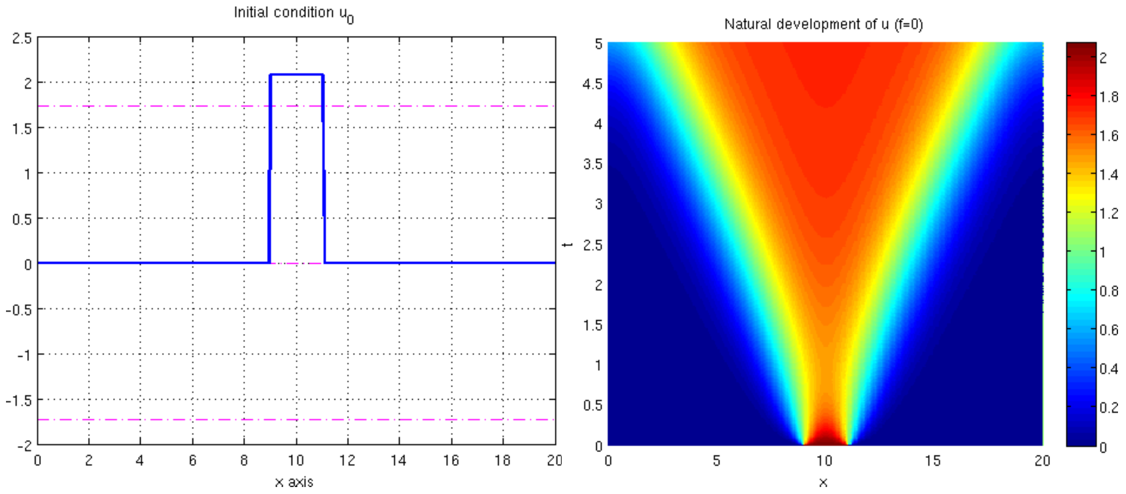


Figure 0.1: Initial state $u_0$ (left) and natural uncontrolled state $u_{\text{nat}}$ (right).

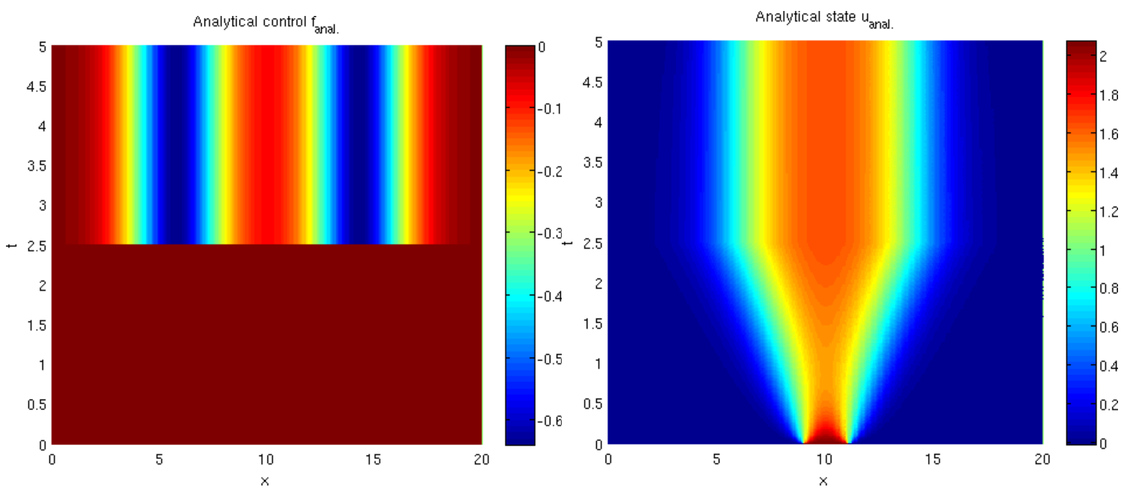


Figure 0.2: Control $f_{\text{stop}}$ (left) and desired state $u_Q$ (right).